MATHEMATICAL VIRUSES

1. PRELIMINARIES ON DIFFERENTIATION

The concept of mathematical viruses is due to David Hestenes in his paper *Mathematical Viruses* in “Clifford Algebras and their Applications in Mathematical Physiscs” edited by A. Micali, R. Boudet and H. Helmstetter (1992). He defines mathematical viruses as preconceptions about the structure, function or method of mathematics which impairs one’s ability to do mathematics. Our endorsing of his concept is not an endorsement of his judgment identifying viruses. We shall speak here of the viruses which affect the understanding of Cartan’s and Kähler’s work. Let us start with some digressions that provide the background for our discussions.

Cartan used the concept of exterior derivative of a tensor-valued form to refer to what nowadays is called the exterior contravariant derivative. Starting with Kähler, one denotes such derivative with the symbol “*d*”. Flanders has formalized the approach to this generalized operator. But most mathematicians would refer to this generalized *d* as exterior derivative because it does not satisfy $d^2 = 0$. Let $\mathbf{e}_i$ be the i$^{\text{th}}$ element in a field of bases, and let $\omega_i^{\ j}$ be the linear part of the connection (the non-linear part is the translation, $d\mathbf{P} = \omega^i \mathbf{e}_i$). With modern notation, Cartan would have written

$$dd\mathbf{e}_i = \Omega_i^{\ j}\mathbf{e}_j = R_i^{\ j}{}_{kl}\,\omega^k \omega^l \mathbf{e}_j, \tag{1}$$

where ω^m can be, in particular, dx^m. He actually wrote $(d\mathbf{e}_i)'$ instead of $dd\mathbf{e}_i$, since the symbol d for Cartan’s “ ' ” was introduced only later, by Kähler. It might be better to write $d(\bar{d}\mathbf{e}_i)$ instead of $dd\mathbf{e}_i$ in order to emphasize that $\bar{d}\mathbf{e}_i$ is not an exact differential form. We do not bother too much about this, however, since, in teleparallelism, $d\mathbf{e}_i$ is a closed form and, therefore, locally exact.

Let *v* be a vector field on a differentiable manifold. We then have:

$$ddv = R_i^{\ j}{}_{kl}\, v^i \omega^k \wedge \omega^l \mathbf{e}_j\,. \tag{2}$$

Hence v^i is acting at each point of the differentiable manifold as a basis of the dual space V^* of the tangent space V at each point in question (see *Elements of the Tensor Calculus* (section 1.11) by A. Lichnerowicz for this view of dual vector spaces). However, using the symbols v^i for the components of v and for the elements of a basis of V^* is not very fortunate, since the two v^i 's that correspond respectively to those two roles have very different derivatives. Hence, let us use the symbols ϕ^i for the elements of the fields of bases of 1-forms (i.e. of linear functions of vectors). They are usually chosen so that, for a given basis $(\mathbf{e}_i)$, they satisfy $\phi^i(\mathbf{e}_j) = \delta^i{}_j$ (also written as $e_j \lrcorner \phi^i = \delta^i_j$). However, this choice is not necessary.

Cartan did not elaborate on the fact that there are three types of indices in $R_i{}^j{}_{kl}$. Kähler was apparently the first one to start writing components of tensor-valued differential forms with three series of indices, one series of superscripts and two series of subscripts. In the case of $R_i{}^j{}_{kl}$, i belongs to a series of subscripts; k and l belong to the other one. We thus have a product of three structures, namely a field of tangent vector spaces, the field of their dual vector spaces and the module of differential 2-forms. One thus may define a (1,1) – tensor-valued differential 2-form $\mathbb{R}$:

$$\mathbb{R} \equiv R_i{}^j{}_{kl}\phi^i e_j \omega^k \wedge \omega^l . \tag{3}$$

The missing symbols for tensor product are understood. $\mathbb{R}$ is at the same time a linear function of tangent vectors (a fact explicit in the combination of the ϕ^i 's) and a function of 2-surfaces (a fact explicit in the combination of the $\omega^k \wedge \omega^l$), meant to be integrated on surfaces.

Notice that we have not assumed that our manifold has a metric structure. If it does, we can replace the basis of the dual space V^* with a basis $\{\mathbf{e}^j\}$ of V called the reciprocal basis of $\{\mathbf{e}_i\}$ and defined by $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$. One can then speak of the 2-tensor valued differential form defined as

$$R_i{}^j{}_{kl}\omega^k \wedge \omega^l \mathbf{e}^i \otimes \mathbf{e}_j . \tag{4}$$

We are dealing here with a product of the space of tangent 2-tensor fields and the module of differential 2-forms, the latter ones being viewed as integrands on surfaces.

Because of the issue of ϕ^i versus $\mathbf{e}^i$, some would find that Eq. (3) has a virus, and others would say that, on the contrary, the virus is in Eq. (4). We do not wish to pick that fight. It is not relevant for the issue of understanding Cartan and Kähler. For both of those authors, as for us, $d\mathbf{e}_i = \omega_i^{\ j}\mathbf{e}_j$, $d\phi^i = \omega_j^i \phi^j$ and $d\mathbf{e}^i = \omega_j^i \mathbf{e}^j$. On the other hand, $d\omega^i$ is independent of connection. Hence, the difference between ϕ^i and $\mathbf{e}_j$ with regards to differentiation is miniscule; but the difference between ϕ^i and ω^i is enormous. ω^i acts on curves on the manifold. ϕ^i acts on vectors in a tangent bundle (called by most cotangent bundle, but we would rather reserve this name for the modules where the animals like $\omega^k \wedge \omega^l$ live). When differentiating (3) or (4), we can introduce the connection through $d\omega^i = \Omega^i + \omega^j \wedge \omega_j^i$, where Ω^i is the torsion. If $\Omega^i = 0$, then $d\omega^i = \omega^j \wedge \omega_j^i$, where ω_j^i now is the Levi-Civita connection. This equation does not mean that $d\omega^i$ depends on the connection. This is rather an equations used to implicitly define the Levi-Civita connection, used together with the statement of metric compatibility.

Remarks:

a) Kähler does not use bases $\mathbf{e}_i$ and ϕ^i explicitly. We use them because it makes the formula of the Kähler calculus far less cumbersome.

b) Cartan, Kähler, Rudin and the authors of this web site refer to the ω^i (and dx^i) as differential forms. For the Bourbakists and most modern mathematicians, the term differential forms denotes the fields of antisymmetric multilinear functions of vectors.

2. VIRUSES RELATED TO THE CALCULUS

2.1 **Bachelor Algebra Virus**. It consists in incorporating everything into just one algebra when two or more of them are involved. For instance, a strain of this virus consists in viewing a curvature (affine or metric) as

$$R_i{}^j{}_{kl}\mathbf{e}^i \otimes \mathbf{e}_j \otimes \mathbf{e}^k \otimes \mathbf{e}^l \ . \tag{5}$$

Consider next

$$d\mathbf{P}\cdot d\mathbf{P} = dx^i\mathbf{e}_i \cdot dx^j\mathbf{e}_j = g_{ij}dx^i dx^j \ , \tag{6}$$

which are familiar equations in the theory of the moving frame. It is understood that the metric is $g_{ij}dx^i \otimes dx^j$. Hence the left hand side of (6) should be written rather as $d\mathbf{P}(\otimes,\cdot)d\mathbf{P}$, where the first symbol in the parenthesis is for the product of (our) differential forms (meaning in this case functions of curves). We are not advocating using the symbol $(\otimes,\cdot)$, but simply advocating awareness of this fact, the rational being as follows. In the Kähler calculus, we have products $(\vee,\otimes)$, where $\vee$ stands for Clifford product of (our) differential forms. Neither $(\otimes,\cdot)$ contains $(\vee,\otimes)$, nor $(\vee,\otimes)$ contains $(\otimes,\cdot)$, in the sense that "$\vee$" (Clifford product) contains "$\wedge$" and "$\cdot$" for vectors, i.e.

$$u \vee v = u \wedge v + u \cdot v \ , \tag{7.1}$$

$$u \wedge v = \frac{1}{2}[u \vee v - v \vee u], \tag{7.2}$$

$$u \cdot v = \frac{1}{2}[u \vee v + v \vee u]. \tag{7.3}$$

One would whish to have some double product that either contains or replaces $(\otimes,\cdot)$ and $(\vee,\otimes)$.

2.2 **The Unisex Virus.** This is a mild virus consisting in that a symbol is used by some author to play one role after another in quick succession. Consider, for instance, the operator

$$A \equiv \zeta^i(x)\frac{\partial}{\partial x^i} \tag{8}$$

for what Cartan denotes as an infinitesimal transformation in *Leçons sur les invariants integraux* [In the modern literature, this is a vector field, but not in Cartan, where vector fields are passive, not operators]. Cartan never defines the action of A on a vector field, thus on a linear function of vector fields. It is clear that Ω here is a differential r – form, meaning a functions of r – surfaces. On the other hand, he defines the evaluating of Ω on A. In the modern literature, this would be equivalent of evaluating an antisymmetric r – linear function of vector fields on just one vector field to obtain an antisymmetric $(r-1)$ - linear function of vector fields. The unisex virus amounts to making the symbols themselves become (in this case) the differential forms, which may represent

(a) the functions of r – surfaces in some cases,

(b) the antisymmetric r – linear functions of vectors in other cases, and

(c) even obtain other meanings in still other cases.

A problem, however, remains if one adopt this view of the symbols as the differential forms themselves, namely that it has exterior and covariant derivatives respectively in cases (a) and (b). For Cartan, Kähler and ourselves, those symbols admit only exterior derivatives. Hence, when differentiating, they are viewed by those authors as integrands, not as antisymmetric functions of vector fields, which have covariant derivatives.

2.3 **The Transmutation Virus.** As we have seen, Kähler must have recognized the need to deal explicitly with the fact the differential forms necessary for his calculus are not the antisymmetric multilinear functions of vector fields. One explains in this way his use of two series of subscripts. He may have been trying to avoid controversy in not providing an explanation for it. That is, however, obvious from his definition by ansatz of his general derivative, comprising exterior, covariant, exterior-covariant, interior and interior-covariant derivatives. Hence, by the standards of Kähler, one should not confuse ϕ^i with ω^i. Doing so constitutes the transmutation virus. A terrible effect of this virus consists in thinking of $d\mathbf{P} \equiv \omega^i \mathbf{e}_i$ as if it were $\phi^i \mathbf{e}_i$. They have different uniquely defined derivatives in the Kähler calculus (also in Cartan's work on differential geometry, though less explicitly so).

Another example of the transmutation virus consists in thinking of the electromagnetic field, $\frac{1}{2} F_{ij} dx^i \wedge dx^j$ (a function of surfaces) as $\frac{1}{2} F_{ij} \phi^i \wedge \phi^j$.

Corresponding to this difference, and as Cartan pointed out, Maxwell's equations do not depend on the connection of spacetime. If Maxwell's equations were viewed as pertaining to $\frac{1}{2}F_{ij}\phi^i \wedge \phi^j$, i.e. as referring to something that refers to each point rather than to a 2-surface, Maxwell's equations would depend on the connection of spacetime (see Cartan's 1924 paper on affine connections, i.e. the second one of his three paper series on those connections).

3. OTHER VIRUSES

3.1 **The Monocurvature Virus**. It consists in ignoring (or writing or speaking) as if manifolds endowed with a Euclidean connection (i.e. metric-compatible affine connection) had just one curvature. Books dealing with Euclidean connections usually fail to make the point that the metric defines a metric curvature, which plays metric roles independently of what the affine connection of the same manifold is. If the affine connection of the space is the Levi-Civita connection, one can think of the curvature as being the symbols themselves, which play then two roles, metric and affine. This, however, has the effect of leaving physicists readers inadvertent of the fact that, when the torsion is not zero, they have two different curvatures to play with, and that they are related to each other through the torsion. They also fail to realize that the original theory of general relativity (1915) does not speak of the affine connection, which did not exist as a concept until two years later. See [LINK] (The Theory > Nay Sayers) for more on this issue.

3.2 **The Riemannitis Virus.** This virus consists in viewing Riemannian geometry as the theory of the invariants of a quadratic differential form $g_{ij}dx^i dx^j$ in n variables x^i with respect to the infinite group of analytic transformations of those variables. Élie Cartan referred to the Riemannian spaces viewed from that perspective as the "false spaces of Riemann" (see page 4 of his 1924 paper *The recent generalizations of the notion of space*.) The reason for doing so is that, as he pointed out in his note of 1922 "On the equations of structure of generalized spaces and the analytic expression of Einstein's tensor", "the ds^2 does not contain all the geometric reality of the space…"

Cartan did not provide a vaccine, meaning a formalization in modern style of the new view of Riemannian and similar geometries that he advocated and described. A vaccine has been provided by R.W. Sharpe in his book *Differential Geometry: Cartan's generalization of Klein's Erlangen program*. He defines a Riemannian geometry on a smooth manifold *M* as a torsion free Euclidean geometry on *M* is a Cartan geometry on *M* modeled on Euclidean space, where a Cartan geometry is a smooth manifold *M* together with a specific Cartan structure, where a Cartan structure … We rather prescribe the Y.H. Clifton vaccine (see the section on pre-Finslerian affine connections in our 1993 paper *Finslerian structures: the Cartan – Clifton method of the moving frame*).

In spite of the present availability of those vaccines the Riemannitis virus has given rise to a plague that infects virtually all physicists and mathematicians who deal with this geometry. It causes a mental block impeding potentially successful efforts to unified the gravitational interaction with the other ones.

3.3 **The Curveism Virus**. This virus is a mutant of the Riemannitis virus. It consists in viewing Finsler geometry as the geometry based on the element of arc

$$ds = F\left(x^1, ..., x^n, dx^1, ..., dx^n\right),$$

where F is positively homogeneous of degree 1 in dx^i. The Finsler spaces so defined are false spaces in the same sense as when Cartan referred to the original Riemannian spaces as false spaces. This virus does not make sick those who work on global geometry or variational problems since they are interested in connection independent results, even though some of the practitioners use connections as a tool to achieve their goals more effectively. The foregoing, curveism – infected definition of Finsler geometry causes blindness since it only allows for connections which are, in some sense or another, canonically determined by the element of arc. In the spirit of Cartan, Finsler geometry should be defined in terms of the structure of a space irrespective of distance considerations. Sharpe did that for Riemannian-like geometries, not for Finslerian-like ones (see the previous subsection, on the Riemanitis virus). Clifton did (see our #5 paper). The key ingredient lies on which bundle is the connection defined, since the same set of frames can be fibrated in more than one way. There are Finsler bundles. There are also Finslerian connections on Riemannian metrics (see our paper #7).