

Of the Exterior Calculus and Relativistic Quantum Mechanics

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January 20, 2019

To Mr. Eric Alterman, for his funding of events promoting the Kähler calculus

Abstract. In 1960-62, E. Kähler developed what looks as a generalization of the exterior calculus, which he based on Clifford rather than exterior algebra [1], [2] and [3]. The role of the exterior derivative, du , was taken by the more comprehensive derivative ∂u ($\equiv dx^\mu \vee d_\mu u$), where “ \vee ” stands for Clifford product. The $d_\mu u$ represents a set of quantities to which he referred as covariant derivative, and for which he gave a long, ad hoc expression. We provide the geometric foundation for this derivative, based on Cartan’s treatment of the structure of a Riemannian differentiable manifold without resort to the concept of the so called affine connections.

Buried at advanced points in his presentations [1], [3] is the implied statement that $\partial u = du + *^{-1}d u*$, the sign at the front of the coderivative term is a matter of whether we include the unit imaginary or not in the definition of Hodge dual, $*$. We extract and put together the pieces of theory that go into his derivation of that statement, which seems to have gone unnoticed in spite of its relevance for a quick understanding of what his “Kähler calculus”.

Kähler produced a most transparent, compelling and clear formulation of relativistic quantum mechanics (RQM) as a virtual concomitant of his calculus. We shall enumerate several of its notable features, which he failed to emphasize. The exterior calculus in Kähler format thus reveals itself as the computational tool for RQM, making the Dirac calculus unnecessary and its difficulties spurious.

1 Introduction

In the preface of his authoritative book “The Dirac Equation”, B. Thaller states:

Perhaps one reason that there are comparatively few books on the Dirac equation is the lack of an unambiguous quantum

mechanical interpretation. Dirac's electron theory seems to remain a theory with no clearly defined range of validity, with peculiarities at its limits which are not completely understood. Indeed, it is not clear whether one should interpret the Dirac equation as a quantum mechanical evolution equation, like the Schrödinger equation for a single particle. The main difficulty with a quantum mechanical one-particle interpretation is the occurrence of states with negative (kinetic) energy. Interaction may cause transitions to negative energy states, so that there is no hope for a stability of matter within that framework. In view of these difficulties R. Jost stated "The unquantized Dirac field has therefore no useful physical interpretation". Despite this verdict we are going to approach these questions in a pragmatic way. A tentative quantum mechanical interpretation will serve as a guiding principle for the mathematical development of the theory. It will turn out that the negative energies anticipate the occurrence of antiparticles, but for the simultaneous description of particles and antiparticles one has to extend the formalism of quantum mechanics. Hence the Dirac theory may be considered a step on the way to understanding quantum field theory.[4]

To speak of the Dirac equation is to speak mainly about relativistic quantum mechanics (RQM). But, for this purpose, a replacement of Dirac's mathematical formalism already exists. It is Kähler calculus. Written in German, it has been under the radar except briefly a few decades ago for highly specialized topics. And it has been overlooked that his superior version of RQM is a virtual concomitant of his calculus; one only needs to let the mathematics speak.

The most outstanding feature of the Kähler calculus (KC) is being the refined, formally adequate representation of the exterior calculus cum coderivative. This differentiation involves Hodge duality, operation which does not belong to exterior algebra but to Clifford algebra. One should do the calculus of differential forms in Clifford rather than exterior context.

The exterior calculus was born in 1899, almost in passing in an E. Cartan paper [5], in the area of differential equations known as Pfaff systems. But it did not get traction for many decades. By his own admission Chern's best paper is from 1944 [6]. In it, he almost apologetically, justifies his use of "the theory of exterior differential forms, instead of the ordinary tensor analysis ... " as a matter of convenience.

In 1960-1962, Kähler produced his calculus (KC) [1], [2] and [3]. In the

last decade of his long life, he returned to this topic [7], not to the topic of Kähler metrics and Kähler manifolds [8], for which he is best known. A similar comment applies to his work on Cartan-Kähler theory of exterior systems[9]. One should perhaps take this as a statement of what he considered to be the most important work of his life, or perhaps the one to which one should pay greatest attention. The aim of this paper is to show that, even though the direct and indirect applications of the exterior calculus are many and important, they still pale in comparison with the fact that, in Kähler form, it yields the magnificent version of RQM of which we speak in the second half of this paper [For indirect applications, see for instance those of Kähler manifolds, as described by Bourguignon [10], and numerous papers on global differential geometry by Chern].

In section 2, we shall give a brief description of the essence of the KC. In Section 3, we give the proof of the equivalence of the coderivative and Kähler's interior derivative. In the interest of brevity, we assume that readers know some exterior calculus and some Clifford algebra. It will not escape their attention why the said proof reaches so far; any differential form can be viewed as a member of both exterior and Clifford algebra.

Section 4 is devoted to enumerating several of the great features of the RQM that emerges as a virtual concomitant of the KC. The exterior calculus in Kähler's version can thus appropriate itself of a RQM without the difficulties of the Dirac theory. The future will be looking down on present mathematicians that speak the language of Gauss, Grassmann, Hilbert, Kähler himself, F. Klein, Riemann and Weyl (to name just a few in a constellation of mathematical luminaries over one and a half centuries) for failing to understand the evolution of a mathematical line of development that started with a Leibnizian prescience, namely his *characteristica geometrica* [11].

2 A Cartan approach to the KC

Kähler introduced in ad hoc manner a concept of covariant derivative of differential forms of tensor valuedness. For present purposes, we specialize that formula to scalar-valuedness. It then reads

$$d_h a_{l_1 \dots l_p} = \frac{\partial}{\partial x^h} a_{l_1 \dots l_p} - \Gamma_{hl_1}^r a_{rl_2 \dots l_p} - \dots - \Gamma_{hl_p}^r a_{l_1 l_2 \dots l_{p-1} r}, \quad (1)$$

where $a_{l_1 \dots l_p}$ stands for $a_{l_1 \dots l_p} dx^{l_1} \wedge dx^{l_2} \wedge \dots \wedge dx^{l_p}$. This author is not responsible for this unfortunate approach and choice of connection.

The KC of scalar-valued differential forms only requires a metric structure. The affine (Euclidean, Lorentzian, etc.) structure is irrelevant. In

1922, just before his paper on theory of affine connections, Cartan derived the differential invariants that characterize a differentiable manifold endowed with a metric.

A summary of his argument follows. He decomposes symmetric, quadratic differential forms as sums of squares

$$ds^2 = \sum^n \epsilon_i (\omega^i)^2, \quad (\epsilon_i = \pm 1). \quad (2)$$

The ω_i 's are linear in the dx^i , but depend not only on the x^j but also on $n(n-1)/2$ parameters u . Exterior differentials of the ω^i 's can be written as

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad (3)$$

where the ω_j^i are linear in the dx and the du . We write down

$$(\omega^i)^\delta = 0 \quad (4)$$

to indicate that there are no du 's in (2).

Cartan develops the consequences of (4) and finds

$$\omega_{ij} + \omega_{ji} = 0. \quad (5)$$

The system of equations (3) and (5) uniquely defines the ω_j^i in the “bundle manifold”, i.e. of the (x, u) , and on sections of this bundle (manifold of the x 's). This system is familiar from the theory of Euclidean connections, also improperly known as metric compatible affine connections, which cease to be affine by virtue of the restriction on the underlying group. But no connection has been used here. We shall write this system as

$$d\omega^i = \omega^j \wedge \alpha_j^i, \quad \alpha_{ij} + \alpha_{ji} = 0, \quad (6)$$

reserving ω_j^i for the actual connection of a manifold, which need not be the Levi-Civita connection. Kähler's derivative ∂ is conceived as

$$\partial u = \omega^\mu \vee d_\mu u = du + \delta u \quad (7)$$

where

$$du = \omega^\mu \wedge d_\mu u, \quad \delta u = \omega^\mu \cdot d_\mu u \quad (8)$$

in the Kähler algebra, i.e. the Clifford algebra determined by

$$\omega^\mu \vee \omega^\nu + \omega^\nu \vee \omega^\mu = 2g^{\mu\nu}, \quad (9)$$

$g^{\mu\nu}$ being the inverse matrix of the $g_{\lambda\rho}$, in turn defined by $ds^2 = g_{\mu\nu}\omega^\mu\omega^\nu$.

It is well known from Clifford algebra that

$$\omega^\mu \wedge \omega^\nu = \frac{1}{2}(\omega^\mu \vee \omega^\nu - \omega^\nu \vee \omega^\mu), \quad (10)$$

and

$$\omega^\mu \cdot \omega^\nu = \frac{1}{2}(\omega^\mu \vee \omega^\nu + \omega^\nu \vee \omega^\mu). \quad (11)$$

Here we are not requiring that the ω^μ 's be orthonormal. We are starting to use Greek indices as we shall reserve Latin indices for 3-space.

The Leibniz distributive rule without alternating signs is assumed. We shall then only need $d_\mu\omega^\nu$ in order to have $d_\mu u$ for any scalar-valued differential form u . On sections of the bundle (i.e. the manifold of the x'_s and $u's''$, the α_μ^ν 's are not independent of the ω^μ 's. We write $\alpha_\mu^\nu = \Gamma_{\mu\lambda}^\nu\omega^\lambda$. Hence

$$d\omega^\mu = \omega^\nu \wedge \alpha_\nu^\mu = \omega^\nu \wedge \Gamma_{\nu\lambda}^\mu\omega^\lambda = \omega^\lambda \wedge (-\Gamma_{\nu\lambda}^\mu\omega^\nu). \quad (12)$$

For $u = \omega^\mu$, the first of equations (8) becomes

$$d\omega^\mu = \omega^\nu \wedge d_\nu\omega^\mu. \quad (13)$$

Comparison of (12) and (13) allows us to obtain two canonical d_λ for ω^μ , namely

$$d_\nu\omega^\mu = \alpha_\nu^\mu, \quad d_\lambda\omega^\mu = -\Gamma_{\nu\lambda}^\mu\omega^\nu. \quad (14)$$

A change of indices without consequence allows us to rewrite the second of Eqs. (14) as:

$$d_\nu\omega^\mu = -\Gamma_{\lambda\nu}^\mu\omega^\lambda. \quad (15)$$

We have here two different covariant derivatives, namely the first of (14) and (15). In terms of coordinate bases, $\Gamma_{\lambda\nu}^\mu = \Gamma_{\nu\lambda}^\mu$. Then

$$d_\nu dx^\mu = -\Gamma_{\lambda\nu}^\mu dx^\lambda = -\Gamma_{\nu\lambda}^\mu dx^\lambda = -\alpha_\nu^\mu, \quad (16)$$

which explicitly shows that the two covariant derivatives are the opposite of each other, in case this was not obvious from the moment that they were introduced. We advance that, for notational compatibility with differential geometry, we choose the option (15). We shall not enter the details but point out at the fact that the standard divergence of a vector field is the same as the interior derivative of a differential 1-form when one use (15).

3 Identification of the covariant derivatives of the Cartan's and Kähler's approaches

Kähler proceeded to simplify Eq. (1), by using that

$$\omega_i^k = \Gamma_{i_j}^k dx^j \quad (17)$$

and obtained

$$d_h u = \frac{\partial u}{\partial x^h} - \omega_h^r \wedge e_r u, \quad (18)$$

where ω_h^r is our α_h^r , where $e_r u$ is $= (dx)_r u$, and where $(dx)_r$ is defined by $g_{sr} dx^r$ (coordinates dx_r do not exist except for rectilinear systems). In Kähler's treatment, formula (18) is as ad hoc as the formula (1) form which he obtained it. The appropriate process to get to (1), if that is what we want, is to continue the process of the previous section and thus obtain (18), as we are about to do.

Let ω^R be the differentiable r -form $\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^r$. Let s be $\leq r$ and let $\omega_{(s)}$ be $(-1)^{s-1} \omega^1 \wedge \dots \wedge \check{\omega}^s \wedge \dots \wedge \omega^r$, where $\check{\omega}^s$ means that we have removed the factor ω^s from ω^R , after making it the first factor in the product. Clearly

$$\omega^R = \omega^s \wedge \omega_{(s)} \quad (\text{no sum}). \quad (19)$$

Hence

$$\omega_s \cdot \omega^R = \omega_s \cdot [\omega^s \wedge \omega_{(s)}] = (\omega_s \cdot \omega^s) \omega_{(s)} + 0 = \omega_{(s)}, \quad (20)$$

if by ω_μ we mean the elements of the basis reciprocal of the ω^ν , i.e. $\omega_\mu \cdot \omega^\nu = \delta_\mu^\nu$. We thus have

$$d_\mu \omega^R = \sum_{s=1}^r d_\mu \omega^s \wedge \omega_{(s)} = - \sum_{s=1}^r \Gamma_{\lambda \mu}^s \omega^\lambda \wedge \omega_{(s)} = - \sum_{\sigma=1}^n \Gamma_{\lambda \mu}^\sigma \omega^\lambda \wedge \omega_{(\Sigma)}, \quad (21)$$

where $\omega_{(\Sigma)}$ is $\omega_{(s)}$ for $\sigma \leq r$ and zero for $\sigma > r$. From (20) and (21), we get

$$d_\mu \omega^R = - \sum_{\sigma=1}^n \Gamma_{\lambda \mu}^\sigma \omega^\lambda \wedge (\omega_\sigma \cdot \omega^R). \quad (22)$$

Formula (22) applies for arbitrary bases of differential 1-forms. If the ω^μ 's are dx^μ 's, the $\Gamma_{\lambda \mu}^\sigma$ are the Christoffel symbols. They satisfy $\Gamma_{\lambda \mu}^\sigma = \Gamma_{\mu \lambda}^\sigma$. Thus

$$d_\mu (dx^1 \wedge \dots \wedge dx^r) = -\alpha_\mu^\sigma \wedge e_\sigma dx^1 \wedge \dots \wedge dx^r. \quad (23)$$

We require $d_\lambda u$ to satisfy the Leibniz rule. Equation (23) and the distributive property of d_λ with respect to addition together yield (18) (in coordinate bases!).

4 The Kähler derivative as sum of exterior derivative and coderivative

It is implicit in Kähler's work that his derivative is the sum of the exterior derivative and the coderivative. But the steps in the argument are scattered over several sections containing a large amount of relevant material. The proof is thus buried in his work, so much of it that one might overlook the point that we are making in this paper.

Let η be the linear operator that acting on differential r-forms, u^R , yields

$$\eta u^R = (-1)^r u^R. \quad (24)$$

Let a a differential 1-form and let A be an arbitrary element of the Clifford algebra. Recall the well known relation

$$aA = a \wedge A + a \cdot A, \quad (25)$$

where

$$a \wedge A = \frac{1}{2}[aA + (\eta A)a] \quad (26)$$

and

$$a \cdot A = \frac{1}{2}[aA - (\eta A)a]. \quad (27)$$

From (27) with $a = dx^\mu$ and $A = v$, we get

$$dx^\mu \vee v = (\eta v) \vee dx^\mu + 2dx^\mu \cdot v, \quad (28)$$

to be used further below.

A differential form, c , is said to be constant if $d_\mu c = 0$. By virtue of the Leibniz rule, we have

$$d_\mu(u \vee c) = (d_\mu u) \vee c. \quad (29)$$

We often use redundant notation (like using the parenthesis in this case) for greater clarity. We have

$$\partial u = du + \delta u \quad (30)$$

where

$$du \equiv dx^\mu \wedge d_\mu u, \quad \delta u \equiv dx^\mu \cdot d_\mu u. \quad (31)$$

We now proceed with the proof. Let z be the unit differential n-form,

$$z = dx^1 \wedge dx^2 \wedge dx^3 \wedge \dots \wedge dx^n. \quad (32)$$

Recall that

$$*u = u \vee z, \quad *^{-1}u = (-1)^{\binom{n}{2}}u \vee z, \quad (33)$$

and

$$*^{-1}d*u = (-1)^{\binom{n}{2}}d(u \vee z) \vee z. \quad (34)$$

We use that z is a constant differential and that

$$dx^\mu \cdot (u \vee v) = (dx^\mu \cdot u) \vee v + \eta u \vee (dx^\mu \cdot v). \quad (35)$$

We replace u and v with $d_\mu u$ and z :

$$\begin{aligned} d(u \vee z) &= dx^\mu \wedge (d_\mu u \vee z) = dx^\mu \vee (d_\mu u \vee z) - dx^\mu \cdot (d_\mu u \vee z) \\ &= \partial u \vee z - \delta u \vee z - (\eta d_\mu u) dx^\mu \vee z. \end{aligned} \quad (36)$$

Taking into account (28), the Clifford product of the last term of (36) by z on the right becomes

$$\begin{aligned} -(\eta d_\mu u) \vee dx^\mu \vee z &= -dx^\mu \vee d_\mu z \vee z + 2(dx^\mu \cdot d_\mu u) \vee z \\ &= -\partial u \vee z + 2\delta u \vee z. \end{aligned} \quad (37)$$

Hence

$$d(u \vee z) = \partial u \vee z - \partial u \vee z - \partial u \vee z + 2\delta u \vee z = \partial u \vee z \quad (38)$$

and, therefore, (34) becomes

$$*^{-1}d*u = (-1)^{\binom{n}{2}}(\delta u \vee z)z = \delta u. \quad (39)$$

We further have, from (30),

$$\partial u = du + *d*u, \quad (40)$$

as we wanted to prove.

5 Kähler version of relativistic quantum mechanics

Kähler's version of relativistic quantum mechanics (RQM) has gone largely unnoticed. Worse yet is the fact that those who have cited his papers on the subject —a rare event in recent decades— appear not to have noticed its great advantages over Dirac's theory. We proceed the document this.

The Kähler equation, to which he deferentially but improperly referred as Dirac’s equation, reads

$$\partial u = au, \tag{41}$$

where a is some input scalar valued differential form and where u is not required to be a spinor but just a member of the Clifford algebra of differential forms. Juxtaposition of symbols (in au but not in ∂u) is here an alternative for the symbol \vee . No valuedness other than scalar is needed for present purposes.

Kähler used equation (41) to solve the hydrogen atom with little “extra effort”. To be sure, one does not reach the fine structure in a couple of steps. But the effort involved lies in the development of rich theory of structural nature, thus useful for other purposes. With the same equation and also almost effortlessly, this author derived from (48) the Pauli-Dirac equation and, in one more page, the Foldy-Wouthuysen Hamiltonian.

In Dirac’s theory, one usually restricts oneself to electromagnetic coupling. As pointed out by Thaller, the range of validity of that equation is not clear. Whereas Kähler considered $\partial u = 0$ to be the equivalent of a Dirac equation, others would not think so. This equation defines (strict) harmonicity, a topic that is not associated with Dirac, except perhaps in some recondite publication. This is to be contrasted with the fact that Kähler gave the title “Integrals of the Dirac equation $\partial u = 0$ in three dimensional Euclidean space” to one of the sections of this 1962 paper. In passing, we shall give here some inklings about unusual applications, thus extended range of validity, of his equation.

The fact that the KC allows us to obtain results in a new, simpler way does not have per se much more than anecdotal evidence. But there is the transcendental feature of (41) that u need to be a spinor, i.e. not belong to an ideal of that algebra. So, it is not an equation for just particles of spin 1/2. It also applies to fields that have lost connection with any specific particle.

Crucial for the important subject of conservation laws is the concept of what he called scalar products of different grades, denoted as $(u, v)_i$, the grade being $n - i$. For $i = 0$, he wrote simply (u, v) . The first Green identity reads

$$d(u, v)_1 = (u, \partial v) + (v, \partial u). \tag{42}$$

If solutions u and v of an equation satisfy that the right hand side of (42) is zero, a conservation law follows. Let overbar denote complex conjugate. Kähler showed that, if $-\eta\bar{a} = a$, and if u and v are solutions of the Kähler

equation with input a , then conservation laws

$$d(u, \eta \bar{v})_1 = 0 \quad (43)$$

follow and, in particular,

$$d(u, \eta \bar{u})_1 = 0. \quad (44)$$

We do not give the definition of $(u, v)_1$, as it would be an unnecessary distraction here. Suffice to mention our use of the notation $\langle u|v \rangle$ and $(u|v)$ for $(u, \eta v)_1$, respectively in the Kähler algebras built upon the modules spanned by (dt, dx^i) and (dx^i) , $i = 1, 2, 3$. Coefficients nevertheless depend on t in both cases. But $\partial/\partial t$ and thus dt will be absent in the second case

Let C be any constant element of the spacetime algebra such that C^2 (i.e. $C \vee C$) equals 1. The $\frac{1}{2}(1 \pm C)$ are two mutually annulling constant idempotents, I^\pm . By virtue of (29),

$$\partial(u \vee I) = dx^\mu \vee d_\mu(u \vee I) = dx^\mu \vee d_\mu u \vee I \equiv \partial u \vee I. \quad (45)$$

Equations (41) and (45) together imply

$$\partial(u \vee I) = \partial u \vee I = au \vee I = a(u \vee I), \quad (46)$$

which shows that the $u \vee I^\pm$ are solutions of the same Kähler equation as the u 's. Because their sum is unity, we have

$$u = u \vee I^+ + u \vee I^- \quad (47)$$

And because they mutually annul, multiplication of (47) respectively by I^+ and I^- uniquely defines ${}^+u$ and ${}^-u$ in

$$u = {}^+u \vee I^+ + {}^-u \vee I^-. \quad (48)$$

Since $(dt)^2 = -1$, we have the constant idempotents

$$\epsilon^\pm = \frac{1}{2}(1 \pm idt). \quad (49)$$

We take exception to not exhibiting the unit imaginary because "i" is of the essence here.

We next assume that the input a of the Kähler equation satisfies $\eta \bar{a} = a$. This is the case in particular for electromagnetic coupling, $-iE_0 + e\omega$, where E_0 is rest mass and where $e = \mp |e|$ is the charge of electron/positron. Kähler uses that

$$u = {}^+u \vee \epsilon^+ + {}^-u \vee \epsilon^-, \quad (50)$$

but not necessarily stationarity. After some calculations, he gets

$$\begin{aligned} \langle u|u \rangle &= \frac{1}{2}(^+u, ^+\bar{u}) + \frac{1}{2}(^+u|\eta^+\bar{u}) \wedge idt \\ &\quad - \frac{1}{2}(^-u, ^-\bar{u}) + \frac{1}{2}(^-u|\eta^-\bar{u}) \wedge idt. \end{aligned} \quad (51)$$

It goes without saying that, if

$$d(u, \eta\bar{u})_1 = d \langle u|u \rangle = 0, \quad (52)$$

a conservation law of the form

$$d(j^{(1)} + j^{(2)}) = 0,$$

follows. The $j^{(1)}$ and $j^{(2)}$ are spacetime currents that come with corresponding densities, $\pm \frac{1}{2}(^\pm u, ^\pm \bar{u})$.

The sign definiteness of the charge densities, which are of identical form except for sign, speaks of the fact that, in Kähler's quantum mechanics, the wave function is about amplitude of charge density, not of probability density. Of course, the Copenhagen interpretation will still work in the situations in which it is applied, but not because of being a basic tenet of this quantum mechanics, where it is not a fundamental but a derived tenet.

Endowed with this interpretation of what the two terms in (50) are when $\eta\bar{a}$ equals a , let us assume stationarity. Kähler then writes u as

$$u = p^+ \vee T^+ + p^- \vee T^-, \quad (53)$$

where

$$T^\pm = e^{-itE/\hbar} \epsilon^\pm. \quad (54)$$

Though Kähler could have considered solutions $p^+ \vee T^+$ and $p^- \vee T^-$ independently of each other, he chose not to do so. There is no need for that as, the equation for u immediately splits into the equations

$$\partial p^\pm \pm \left(\frac{E}{\hbar} + \beta \right) \vee \eta p^\pm - \alpha \vee p^\pm = 0 \quad (55)$$

after decomposing a as $\alpha + \beta \vee idt$. The two equations differ only by the sign of the second term. Notice that we have two equations for the same E , not

$$\partial p^\pm + \left(\pm \frac{E}{\hbar} + \beta \right) \vee \eta p^\pm - \alpha \vee p^\pm = 0. \quad (\text{NOT!})$$

We emphasize that (55) is for coupling more general than electromagnetic, in which case they would be for positions and electrons respectively. Notice in (57) that they are associated with the same sign of the energy. Notice further that there is neither room nor need to eliminate small components of the wave function as if they belonged to an antiparticle contamination of the wave function. As is well known, that is the case in Dirac's theory.

What we have said so far is part of the twelve pages on RQM in his paper [3], part of which is devoted to the fine structure of the H atom. In a previous paper [2], he had solved the same problem starting with proper solutions for energy and momentum. For that purpose, he defined idempotents τ^\pm :

$$\tau^\pm = \frac{1}{2}(1 \pm idxdy). \quad (56)$$

They add up to unity, kill each other and commute with ϵ^* . This allows him to write

$$u = {}^+u^+ \vee \tau^+ \vee \epsilon^+ + {}^+u^- \vee \tau^- \vee \epsilon^+ + {}^-u^+ \vee \tau^+ \vee \epsilon^- + {}^-u^- \vee \tau^- \vee \epsilon^-, \quad (57)$$

the ${}^\pm u^*$ being all well defined in terms of u :

$${}^\pm u^* = u \vee \tau^* \vee \epsilon^\pm. \quad (58)$$

One can associate τ^\pm with spin/chirality in the same way as ϵ^\pm is associated with energy/charge. Kähler then assumed central fields, apparently so that the system of a cylindrically symmetric particle in a field will not lose this symmetry by virtue of the non-cylindrically symmetric field. He wrote the ansatz

$$u = e^{im\phi - iEt/\hbar} p \vee \tau^\pm \vee \epsilon^*, \quad (59)$$

where p depends on $(\rho, z, d\rho, dz)$, but not on $(t, \phi, dt, d\phi)$, where ϕ is angular cylindrical coordinate and where m is angular momentum. In view of the combined (57) and (59) equations, it is clear that both positrons and electrons fit together at the same time in the Kähler equation.

Another great result of Kähler version of RQM is his treatment of angular momentum, which he approaches from a perspective of Lie differentiation. But this differentiation is not what one would expect, as vector fields and their flows do not enter his concept of Lie differentiation (unless one defines vector fields as $\partial/\partial x^\mu$ operators and their linear combinations, which neither Cartan nor Kähler do). The major though not unique result of Kähler's treatment of angular momentum, $\partial/\partial\phi$, is that both orbital and spin angular momentum are unified ab initio, actually before being born, the meaning of which we shall explain below.

Kähler refers to the operators (which cyclic i, j, k)

$$X_i = x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j}, \quad (60)$$

as Lie operators (see formula 22.2 of [3]). All the three X_i are of the form $\partial/\partial\phi_i$, where ϕ_i is azimuthal coordinate with respect to the three axes. With $w_i \equiv dx^j \wedge dx^k$, the action of X_i on u is

$$X_i u = x^j \frac{\partial u}{\partial x^k} - x^k \frac{\partial u}{\partial x^j} + \frac{1}{2} w_i \vee u - \frac{1}{2} u \vee w_i. \quad (61)$$

The argument leading to this equation from “first principles” is given in section 33 of [1].

We proceed to illustrate how the argument goes. Given operator X ,

$$X = \alpha^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}. \quad (62)$$

Kähler introduces the differential system

$$\frac{dx^i}{dy^n} = \alpha^i(x^1, \dots, x^n), \quad (63)$$

which is a familiar system from classical mechanics, where y^n is time, t . The reason for using the symbol y^n is that the $n - 1$ independent constants of the motion not additive to y^n together with y^n constitutes a new coordinate system, y^i , and thus

$$x^i = x^i(y^1, \dots, y^n). \quad (64)$$

In terms of the y coordinate system X reduces to $\partial u/\partial y^n$. Kähler then computes $\partial u/\partial y^n$ with $u = a_R dx^R$ and obtains

$$\frac{\partial u}{\partial y^n} = (X a_R) dx^R + d\alpha^i \wedge e_i u. \quad (65)$$

He has resorted to coordinates (y^i) because the different terms on the right of (62) correspond to different conditions. So application of the sum is not equivalent to sum of the different partial derivatives. On the other hand, $\partial u/\partial y^n$ is just one partial derivative, not a sum of them. It is clear that we have

$$\frac{\partial u}{\partial y^n} = \alpha^i \frac{\partial u}{\partial x^i} + d\alpha^i \wedge e_i u \quad (66)$$

for arbitrary differential forms u .

Kähler refers to the right hand side of (66) as the Lie derivative of u . But, by his argument, it is simply the partial derivative, $\partial u/\partial y^n$. One does not need to explicitly compute y^n . Equation (66) directly gives as its form in term of the original coordinate system. A simple example will help with another remark.

The third component of angular momentum is $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Two (of infinite) coordinate system containing ϕ (i.e. the corresponding y^n) are the spherical and cylindrical ones. ϕ is determined by its coordinate line, which is independent of the other coordinates, r and θ or ρ and z . Hence $\partial/\partial y^n$ is determined neither by a specific allowed (y^i) system, nor by what coordinate system we choose to express u (Cartesian coordinates in our example).

With respect to the unification of the two types of angular momentum, orbital and intrinsic, consider the following. The terms in (66) are not the same in the x and x' coordinate systems (This is remedied by the use of covariant derivatives). Only their sum is. One can add and subtract the appropriate quantity to the respective terms, so that the sum is unchanged and the two terms on the right are covariant. (66) then becomes

$$Xu = \alpha^l d_l u + (d\alpha)^l \wedge e_l u. \quad (67)$$

Kähler uses the symbol $(d\alpha)^i$ to represent what may be viewed as components as a vector of the differential form $d(\alpha^i e)$ under the Levi-Civita connection (This remark is only for identification purposes of what quantities are involved; the manifold may not even be endowed with a connection, i.e. a rule to compare vectors in two tangent spaces).

For X_l , the first (second) term in (67) will yield the first two terms (respectively third and fourth) terms in (64). They represent orbital and intrinsic angular momenta. They are “entangled” in (66) before they became individually meaningful and interpreted as angular momentum concepts in (64). One could not have deeper unification than that.

Equation (67) involves only the exterior product. Kähler developed this equation further when the metric does not depend on the coordinate y^n . The Clifford product emerges in the expression for the Lie derivative. He further specializes it to spacetime. That is how equation (64) results.

Kähler defines total angular momentum not as a vector or bivector operator, but through

$$(K + 1)u = \sum_{i=1}^3 X_i u \vee w^i. \quad (68)$$

The reason for using $k + 1$ on the left rather than just M (i.e. $M = K + 1$)

is that, as he proves

$$K(K + 1) = -(X_1^2 + X_2^2 + X_3^2). \quad (69)$$

One should not worry about the minus sign. It would not appear if we had multiply by i in the definition of X^i . Notice that we could have written the right hand side of (68) as

$$\frac{\partial u}{\partial \phi^i} dx^j dx^k \quad (70)$$

where ϕ^i pertains to the plane (x^j, x^k) . Hence total angular momentum is the differential 2-form operator whose components are the $\partial/\partial\phi^i$. This remark is helpful in understanding the presence of the first exponential and the factor τ in (59).

We have devoted a lot of text to Lie differentiation and angular momentum. It is the price we have to pay for illustrating the many subtleties connected with these concepts. More enticing possibilities are intimated in the next section.

6 Loose ends and the mathematical owner of relativistic quantum mechanics

In papers [1], [2] and [3], Kähler gave not a name to his underlying algebra. In an additional paper of 1964 [12], he summarized results from the aforementioned papers and gave the name of Clifford to the algebra he had been using, and again in 1992 [7].

We have only focussed on the relation between the coderivative and the interior derivative. We take it for granted that readers know how to relate exterior products to Clifford products and vice versa. Kähler's formula (9.1) of [3] expresses the Clifford product of two arbitrary elements of the algebra in terms of exterior products. In particular, one can make one of the two factors be the unity.

For the reverse process, i.e. to show that the exterior product can be written in terms of Clifford products, suffice to show that such is the case for the product of two monomials,

$$\begin{aligned} u \wedge v &= (adx^1 \wedge \dots \wedge dx^r) \wedge (bdx^{r+1} \wedge \dots \wedge dx^s) \\ &= abdx^1 \dots \wedge dx^r \wedge dx^{r+1} \dots \wedge dx^s. \end{aligned} \quad (71)$$

The proof then goes as follows. Let $Y^i \equiv dx^{i+1} \wedge \dots \wedge dx^s$. Then

$$dx^1 \wedge Y^1 = \frac{1}{2}[dx^1 \vee Y^1 + (\eta Y^1) \wedge dx^1]. \quad (72)$$

One can express Y^1 as

$$\frac{1}{2}[dx^2 \vee Y^2 + (\eta Y^2) \vee dx^2] \quad (73)$$

and proceed similarly with ηY^2 , and then Y^3 , etc. This should be enough to justify our claim.

There is a Kähler calculus more comprehensive than the one we have discussed. It deals with tensor-valued differential forms. Kähler considers the curvature as a tensor-valued differential 2-form. But Euclidean curvatures are bivector-valued differential 2-forms, as Cartan already stated [13]. Bivectors are not antisymmetric tensors since the tensor product of antisymmetric tensors is not an antisymmetric tensor in general. Curvatures are members of quotient algebras (not subalgebras) of the general tensor algebra. We do not find tensor-valuedness to be interesting at all.

There is, however, a very interesting point in Kähler's dealing with tensor-valued differential forms. Their components have three series of indices, two of which are for subscripts. One of these two are for covariant tensors, whether antisymmetric or not. The other one is for differential forms viewed as functions of r-surfaces. For Cartan and Kähler, differential forms are not antisymmetric r-linear functions of vectors. The differentiation of scalar-valued differential forms is determined by the Christoffel symbols. The differentiation of linear functions of vectors is determined by the connection, whose components will not be given by those symbols when there is torsion.

Kähler's version of RQM is a concomitant of his calculus, which, in turn, is nothing but the exterior calculus cum coderivative reformulated so that the underlying algebra is manifestly Clifford algebra, rather than exterior algebra that is complemented with non-exterior concepts like Hodge duality and coderivative. Hence (the Kähler version of) RQM, which we have just shown to be superior to Dirac's, may be said to be owned by the calculus of scalar-valued differential forms. The Dirac calculus, a magnificent achievement of the first third of the twentieth century, is nowadays unnecessary and should be abandoned.

7 Beyond the relation between the exterior calculus and relativistic quantum mechanics

At present, RQM has not a well defined boundary with quantum field theory. The latter can be considered as an extension of the former, but one may

argue that is not natural nor canonical. There are topics which some authors consider as pertaining to RQM and that other authors consider as pertaining to its operator based extension. And there is also the S matrix theoretical alternative to quantum field theory.

Because of its unparalleled contribution to quantum physics since 1948, let us mention that Schwinger finds problems with quantum field theory that his proposed source theory does not have [14]. He also tells us how sources imitate but supersede S matrix theory, *ibid*. Sources have much of the flavor of differential forms, but they still are an ad hoc construction. It is a concomitant of the main result of this paper that, given the relation between the exterior calculus and RQM, one should look for an extension of RQM in the extension of the exterior calculus.

The natural extension of the exterior calculus is differential geometry, which Cartan and many differential geometers view as the exterior calculus of vector-valued forms (the bivector valuedness comes in the wash when the manifold is endowed with a metric). The natural extension of scalar-valuedness then is vector-valuedness, and some algebra built upon the module of vector fields. This takes place very simply through the replacement of the unit imaginary with “mirror elements in the tangent algebra of differential forms in the idempotents. Thus $idxdy$ should be replaced with $\mathbf{ij}dx dy$. For further details, see a series of papers posted in arXiv, where I have started the replacement of the unit imaginary with elements of a tangent Clifford algebra (Type Jose G. Vargas on the right hand corner of the arXiv’s main page and ignore the entries with multiple authorship), We now give an inkling of what to gain with such a replacement.

The two differential forms idt and $idxdy$ give rise to the eight idempotents ϵ^\pm , τ^\pm and $\epsilon^\pm\tau^*$. With one more square root of one, we could build idempotents ϵ^\pm , τ^\pm , λ^\pm , $\epsilon^\pm\tau^*$, $\epsilon^\pm\lambda^*$, $\tau^\pm\lambda^*$ and $\epsilon^\pm\tau^*\lambda^{**}$. There are 2^3 of them just of the type $\epsilon^\pm\tau^*\lambda^{**}$. What could the λ^\pm be? We skip considering $(1/2)(1 \pm idydz)$ since the difference with τ^\pm is just a choice of coordinates. Let us notice in passing that the $(1/2)(1 \pm idydz)$ do not commute with the τ^\pm , but $(1/2)(1 \pm \mathbf{ij}dx dy)$ and $(1/2)(1 \pm \mathbf{jk}dy dz)$.

Consider also idempotents of the type $\lambda_i^\pm = (1/2)(1 \pm \mathbf{a}_i dx^i)$, with no sum over repeated indices. The three λ_i^\pm commute among themselves and with some of the idempotents previously considered. The issue arises of what the presence or absence of commutativity implies.

Finally, let $\epsilon_1^\pm, \epsilon_2^\pm, \epsilon_3^\pm, \dots, \epsilon_r^\pm$ be “monary”, i. e not products of other ones. We can perform all sorts of decomposition of the unity in terms of

them. Thus, for instance,

$$1 = \epsilon_1^+ + \epsilon_2^- = \epsilon_1^+ + \epsilon_2^- \epsilon_5^+ + \epsilon_2^- \epsilon_5^- = \epsilon_1^+ + (\epsilon_2^- \epsilon_5^+ \epsilon_6^+ + \epsilon_2^- \epsilon_5^- \epsilon_6^-) + \epsilon_2^- \epsilon_5^- = \dots \quad (74)$$

where we have introduced parenthesis for greater clarity. If particles are associated with idempotents, decompositions of this type are a golden rule for creating all sorts of plausible particle reactions. We could refer to this mechanism of particle stoichiometry.

That is just an example of what a KC of Clifford-valued differential forms (i.e. members of the tensor product of two Clifford algebras) could do! If you have contrasting ideas, it all will be for the benefit of mathematics.

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