The Meaning of Wave Functions and the Copenhagen Misinterpretation

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ABSTRACT

Following the lead of the great mathematician Erich Kähler (1906-2000), we use the term "Dirac equation" to refer to his generalization of the standard one known by this name. He used his exterior-interior calculus of differential forms and did not confine himself to members of ideals, conventional spinors in particular.

In this paper, and like he did, we derive the conservation law as an immediate consequence of Green's first identity. Unlike Kähler, we separate in the argument what is specific to the Dirac equation with electromagnetic coupling from what is the spinoral development of that identity. We also reorder the presentation so that the punchline becomes obvious...the wave function is amplitude of a quantity which shows up in physics with both signs, not like probability but like charge. The Copenhagen interpretation is just a heuristic (not fundamental) tenet of quantum mechanics.

1 Introduction

The Copenhagen interpretation of the wave function is generally viewed as a fundamental tenet of Quantum Mechanics (QM). In this paper, we show that if we use the calculus of differential forms (difforms), an interpretation follows from Green's Formula. The Copenhagen interpretation then simply is a pragmatic, reliable recipe for explaining experimental results. However, a basic tenet of QM it is not.

For a preliminary understanding of how this paper achieves its objectives, consider the expression

$$\frac{1}{2}\{{}^{+}u,{}^{+}\overline{u}\}+\frac{1}{2}\{{}^{+}u,\eta^{+}\overline{u}\}\wedge icdt-\frac{1}{2}\{{}^{-}u,{}^{-}\overline{u}\}+\frac{1}{2}\{{}^{-}u,\eta^{--}\overline{u}\}\wedge icdt \qquad (1)$$

derived by Kähler for one side of Green's 1st Identity (G1). Here, u is the wave function, ${}^{+}u$ and ${}^{-}u$ are its spinorial components, the overbar notation

represents the complex conjugate of these quantities, and where $\{^{+}u,^{+}\overline{u}\}$ is never negative and $-\frac{1}{2}\{^{-}u,^{-}\overline{u}\}$ is never positive. We shall see that equating this expression to zero states a conservation law, which, in the notation of the vector calculus reads

$$0 = \frac{\partial \rho_1}{\partial t} + div\ \mathbf{J}_1 + \frac{\partial \rho_2}{\partial t} + div\ \mathbf{J}_2. \tag{2}$$

All four of ρ_1, $\rho_{2,}$ $\mathbf{J}_1$, and $\mathbf{J}_2$ are given in terms of the wave function u. For the moment, we need only say that this equation is not solely specific to QM, but also to Lorentz space mathematics and that u can be, but need not be, the wave function.

The content of this paper is a rewrite of part of the material in [1], reordered to separate what is simply spacetime calculus from the role of the wave function in the argument. Kähler arrived at virtually the same results, but they are elucidated here.

In section 2, we deal with G1 (also named Green's formula), as everything in this paper revolves around it. In section 3, we extend the concept of current, showing that each spacetime difform has a current associated with it, which sometimes gives rise to a conserved quantity. In section 4, we obtain the spinorial form of the left-hand side of G1. In section 5, we show the conditions that result in zeros, thus yielding a conservation law. In section 6, we show how these conditions are met by the electromagnetic coupling.

2 On Green's formula, essential for the interpretation of the wave function

This section is about summarizing, refocusing, and extracting from the section "Scalar products" of paper [1] what is most important for our purposes. There, we find G1 as expressed in the Kähler Calculus (KC):

$$d(u,\ v)_1 = (u,\ \partial v) + (v,\ \partial u). \tag{3}$$

The symbol d stands as usual for exterior differentiation, and the symbol ∂ (Kähler uses δ instead) is the operator that plays in the KC the same role as the Dirac operator plays in his Calculus.

Kähler writes difforms u in manifolds of dimension n as

$$u = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1 \ldots i_p}\ dx^{i_1} \wedge \ldots \wedge\ dx^{i_p}, \tag{4}$$

which is typical in the literature of exterior algebra. The factor $\frac{1}{p!}$ can be dispensed with if, instead of summing over all permutations of the indices, we sum only over elements of bases of difforms. This modification still allows us to use Einstein's repeated indices summation convention.

Kähler uses the term scalar product of dimension $n-p$ to refer to

$$(u,\ v)_p \equiv e_{i_1} \wedge ... \wedge\ e_{i_p}(dx^{i_1} \vee ... \vee dx^{i_p} \vee u,\ v) \tag{5}$$

where

$$(u,\ v) = (\zeta u \vee v) \wedge z, \tag{6}$$

ζu is the anti-involution operator, $\vee$ is the Clifford product, and z is the unit difform of grade n. Notice that taking the right exterior product by z picks only the term of grade zero in the expansion of $\zeta u \vee v$ in different grades. The operator e_i acting on u uniquely defines u' and u'' in the equality

$$u = dx^i \wedge u' + u'' \qquad\qquad i = 1, ...n \tag{7}$$

if one demands that neither of the two contain dx^i in their expansions. The operator $e_{i_1} \wedge ... \wedge\ e_{i_p}$ first removes the factor dx^{i_p}. Then, $e_{i_{p-1}}$ removes the factor $dx^{i_{p-1}}$, and so on. If any of the factors $dx^{i_1}, dx^{i_2}, ..., dx^{i_p}$is missing in u, then $e_{i_1} \wedge ... \wedge\ e_{i_p} u = 0$.

We shall be interested only in the scalar products $(u,\ v)$ and $(u,\ v)_1$ defined by Kähler, of respective grades n and $n-1$, regardless of the grades (inhomogeneous in general) of u and v. Of course, such products will often be zero. Because of its relevance, we explicitly write $(u,\ v)_1$ as

$$(u,\ v)_1 = e_i(dx^i \vee u, v) = (\zeta u \vee dx^i \vee v)_0\ e_i z. \tag{8}$$

$()_0$ stands for the $0-$form component of $()$, and z stands for the unit differential $n-$form.

At a key point in his argument, Kähler reserves the notations $(\ ,\ )$ and $(\ ,\ )_1$ for the defined products of spacetime difforms. They are thus of grades 4 and 3. He uses the notation $\{\ ,\}$ and $\{\ ,\}_1$for the similarly defined concepts in $3-$space, thus of respective grades 3 and 2, with the proviso that he, however, lets the coefficients depend on t (though not on dt–i.e., using t as a parameter). Notice that once ∂u and ∂v are given, the right-hand side of Eq. (3) is just a matter of algebra, as also is $(u,\ v)_1$ .The left-hand side, i.e. $d\,(u,\ v)_1$ is primarily about calculus, namely the computation of an exterior differential. That is a key issue for our work in the future on an equation more relevant than Dirac's, with the latter then assuming a subordinate role.

3 The spinorial representation of spacetime difforms

In spacetime, we shall refer to differential 3−forms as currents. In this sense, $(u, v)_1$ is a current. Also in spacetime, we shall use the term spinorial decomposition (of any difform u) for the right-hand side of

$$u = {}^{+}u \vee \epsilon^{+} + {}^{-}u \vee \epsilon^{-}, \tag{9}$$

where the $\epsilon^{\pm}$ are the mutually-annulling idempotents

$$\epsilon^{\pm} = \frac{1}{2}\left(1 \mp idt\right), \qquad \epsilon^{\pm}\epsilon^{\mp} = 0, \qquad \epsilon^{\pm}\epsilon^{\pm} = 1 \tag{10}$$

with $c = 1$ in (1) and where we use juxtaposition of symbols as an alternative way of designating their Clifford product. When they do not contain a dt factor, both ${}^{+}u$ and ${}^{-}u$ are uniquely defined. When dt is present, we can always move it next to $\epsilon^{\pm}$, where it will be absorbed:

$$dt\left(1 \mp idt\right) = \left(dt \pm i\right) = \pm i\left(1 \mp dt\right) \tag{11}$$

We are now interested in the left-hand side of (3), specifically the spinorial form of $(u,\, v)_1$. We have

$$\begin{aligned}(u,\ v)_1 &= \left({}^{+}u \vee \epsilon^{+} + {}^{-}u \vee \epsilon^{-},\ {}^{+}v \vee \epsilon^{+} + {}^{-}v \vee \epsilon^{-}\right)_1 \\ &= \left({}^{+}u \vee \epsilon^{+},\ {}^{+}v \vee \epsilon^{+}\right)_1 + \left({}^{+}u \vee \epsilon^{+},\ {}^{-}v \vee \epsilon^{-}\right)_1 \\ &\quad + \left({}^{-}u \vee \epsilon^{-},\ {}^{+}v \vee \epsilon^{+}\right)_1 + \left({}^{-}u \vee \epsilon^{-},\ {}^{-}v \vee \epsilon^{-}\right)_1.\end{aligned} \tag{12}$$

Note that the middle two of the four terms on the right-hand side of (12) are zero. Thus, for example,

$$\begin{aligned}\left({}^{\pm}u \vee \epsilon^{\pm},\ {}^{\mp}v \vee \epsilon^{\mp}\right)_1 &= \left({}^{\pm}u \vee \epsilon^{\pm} \vee \epsilon^{\pm},\ {}^{\mp}v \vee \epsilon^{\mp}\right)_1 \\ &= \left({}^{\pm}u \vee \epsilon^{\pm},\ {}^{\mp}v \vee \epsilon^{\mp} \vee \zeta\epsilon^{\pm}\right)_1,\end{aligned} \tag{13}$$

where we have used that $\epsilon^{+} \vee \epsilon^{+} = 1$ and $\epsilon^{-} \vee \epsilon^{-} = 1$. For the last step in (13), we have used a property of Kähler's scalar product that allows us to move a trailing $\epsilon^{\pm}$ from the left part of any $(\ ,\ )$ to the end of the second part of that same $(\ ,\ )$ product by first prepending it by a ζ operator. See Kähler's original paper [1]. We further have $\epsilon^{-} \vee \zeta\epsilon^{+} = \epsilon^{-} \vee \epsilon^{+} = 0$. Hence,

the second and third terms of the right-hand side of (12) are zero. It follows that

$$(u,v)_1 = ({}^+u \vee \epsilon^+,\ {}^+v \vee \epsilon^+)_1 + ({}^-u \vee \epsilon^-,\ {}^-v \vee \epsilon^-)_1, \tag{14}$$

with both terms on the right-hand side being here of the type $(p \vee \epsilon^{\pm},\, q \vee \epsilon^{\pm})_1$ where p and q are spatial differentials.

Equation (14) is very significant since this is a decomposition into terms from two different, complementary ideals. We want to look at all three terms in (14) as currents, non-conserved in general.

4 Currents and Invariants

The basic ideas here are from the exterior calculus. We follow E. Cartan's book "Leçons sur les Invariants Integraux"[2]. Let $\rho(t, \widetilde{x})$ denote the volume density of a scalar quantity. The wiggle $\widetilde{}$ on the x is meant to signify $(x^1, ..., x^n)$. As in the exterior calculus, we use the term "3-current" to refer to a spacetime differential $3-form$ that contains a mixture of dt and dx^i's. We also use the term "density" to refer to the differential $3-form$ $\rho(t, \widetilde{x})$ $dx^1 \wedge dx^2 \wedge dx^3$, which contains only dx^i's. The integral

$$\int_V \rho(t, \widetilde{x})\ dx^1 \wedge dx^2 \wedge dx^3 \tag{15}$$

is taken at an instant in time. If this integral is time-dependent, it represents a conserved quantity on volume V and is said to be a Poincaré integral invariant. Cartan relaxed the requirement of integrating at a point in time by replacing $\rho\ dx^1 \wedge dx^2 \wedge dx^3$ with the integrand

$$\rho \cdot (\ dx^1 - u^1 dt) \wedge (dx^2 - u^2 dt) \wedge (dx^3 - u^3 dt), \tag{16}$$

where u^1, u^2, and u^3 are the corresponding coordinate velocities.

Upon development of this expression, we get

$$j \equiv \rho\ dx^1 \wedge dx^2 \wedge dx^3 - \rho\ u^i dt \wedge dx^j \wedge dx^k \tag{17}$$

with summation over the even cyclic permutations. One says that (17) is a Cartan differential invariant $\int_{\mathfrak{V}} j$ taken over a spacetime 3-volume $\mathfrak{V}$ is a constant q. We shall use the term $4-current$ to refer to any differential $3-form$ in spacetime that contains at least one dt factor. If its exterior

derivative is zero, we say that it is a conserved current. In the language of the vector calculus, this conservation takes the form

$$\frac{\partial \rho}{\partial t} + div\ j = 0, \tag{18}$$

where ρ and the coefficients of j are the components of the $4-currents\ j$ defined in (17).

Given any spacetime differential $3-form\ u$, we can consider it as a current by writing it as

$$u = dt \wedge u' + u'', \tag{19}$$

with u' and u'' appropriately separating those terms that contain dt as a factor from those that do not. We shall refer to u'' as a density and to u' as a $3-current$. From them, one can compute the components u^i, or "internal velocities," by comparison of (17) and (19).

Notice that no reference has yet been made to electrodynamics, except for using familiar symbols and names. Also, we are not implying that u is an invariant, unless $du = 0$. We have thus defined an (in general) non-conserved current for any spacetime differential $3-form$ or $4-current$. Of course, this is not of great interest if u' or u'' are zero.

5 The spinoral representation of currents

In this section, we focus exclusively on the right-hand side of the G1 in the form taken in (14). We will now use capitals, i.e.

$$({}^{+}U \vee \epsilon^{+},\ {}^{+}V \vee \epsilon^{+})_1 + ({}^{-}U \vee \epsilon^{-},\ {}^{-}V \vee \epsilon^{-})_1 \tag{20}$$

in order to avoid confusion, as we now explain with two reasons.

First, we use capitals when we are referring to pairs $(U,\ V)$ that emerge in the evaluation of a particular pair $(u,\ v)$, as we shall see in the following paragraphs. Second, Kähler specializes (u, v) to $(v, \eta\overline{u})$ at a much earlier stage than we do, as we are simultaneously preparing the ground for developments in papers to follow and which he did not consider. (Here, η is the main involution operator in the Clifford algebra.) If we had used lowercase where we now use uppercase, this might have confused readers when comparing our formulas to Kähler's.

The two terms in (20) are independent and thus can be developed separately to the very end, using the same process in both cases. Hence, following

Kähler, we develop them together for more economical treatment. That is why he uses the terms p and q to cover both terms. We use P and Q for the same reason. Both terms in (20) are of the type $(P \vee \epsilon^{\pm}, Q \vee \epsilon^{\pm})_1$, where P and Q are space differentials. For any pair of them, we have

$$(P \vee \epsilon^{\pm},\ Q \vee \epsilon^{\pm})_1 = \frac{1}{4}[(P,\ Q)_1\ \mp (P,\ Q \vee idt)_1 \mp (P \vee idt,\ Q)_1 + (P \vee idt,\ Q \vee idt)_1], \tag{21}$$

which follows by simply replacing $\epsilon^{\pm}$ with their definitions. The first and fourth term on the right of (21) are equal to each other, and so are the second and third ones. We thus obtain

$$(P \vee \epsilon^{\pm}, Q \vee \epsilon^{\pm})_1 = \frac{1}{2}(P, Q)_1 \mp \frac{1}{2}(P, Q \vee idt). \tag{22}$$

From the definition of products $(\ ,\ )_1$ in (5), we get

$$(P,\ Q)_1 = e_\mu (dx^\mu \vee P,\ Q) = (\zeta P \vee dx^\mu \vee Q)_0 e_\mu z, \tag{23}$$

where we replaced the index i (with μ) in the x^i of the original [1]. Here, $dx^\mu = dt$ when $\mu = 0$ and $dx^\mu = dx^k$ for $k = 1, 2, 3$. As for z, it equals $w \wedge idt$, where $w = dx^1 \wedge dx^2 \wedge dx^3$. Since neither P nor Q contain dt, neither ζP nor Q in $(\zeta P \vee dt \vee Q)_0$ cancels the dt factor, so that $\zeta P \vee dt \vee Q$ does not have a differential $0 - form$ part. Thus

$$(P,\ Q)_1 = (\zeta P \vee dx^k \vee Q)_0 e_k w \wedge idt = \{P,\ Q\}_1 \wedge idt \tag{24}$$

with the meaning of $\{\ ,\}_1$given in Section 2, now for $3 - space$.

Similarly, we now deal with $(P,\ Q \vee it)_1$ for use in (22).

$$(P,\ Q \vee it)_1 = e_\mu (\zeta P \vee dx^\mu \vee Q \vee idt) \wedge z = (\zeta P \vee dt \vee Q \vee idt)_0 e_t z \tag{25}$$

The terms where $\mu = k$ (i.e. spatial) in (25) are null because P and Q are also spatial and we need dx^μ to be dt in order to cancel the idt factor so that the contents of the parentheses can have a $0 - form$ part. Thus

$$(P,\ Q \vee it)_1 = (\zeta P \vee dt \vee Q \vee idt)_0 e_t z. \tag{26}$$

We next use that

$$Q \vee idt = idt \vee \eta Q, \tag{27}$$

and that

$$e_t z = e_t(w \vee idt) = -ie_t(dt \vee w) = -ie_t(dt \wedge w) = -iw. \tag{28}$$

We thus further have

$$(P,\ Q \vee it)_1 = (\zeta P \vee dt \vee idt \vee \eta Q)_0(-iw) = -(\zeta P \vee \eta Q)_0 w, \tag{29}$$

which, being a spatial form, the last member of this set of equations can be written as $-\{P,\ \eta Q\}$ and therefore,

$$(P,\ Q \vee it)_1 = -\{P,\ \eta Q\}. \tag{30}$$

One should now use (24) and (30) on the right-hand side of (22) after first replacing the pair P and Q with ${}^+u$ and ${}^+v$, and then the same pair with ${}^-u$ and ${}^-v$. For the first case, we have, from (30) and 24,

$$\frac{1}{2}\left\{{}^+u,\ \eta {}^+v\right\} + \frac{1}{2}\left\{{}^+u,\ {}^+v\right\}_1 \wedge idt. \tag{31}$$

We similarly have, for the second pair,

$$-\frac{1}{2}\left\{{}^-u,\ \eta {}^-v\right\} + \frac{1}{2}\left\{{}^-u,\ {}^-v\right\}_1 \wedge idt. \tag{32}$$

The sum of these two expressions is the same as the right-hand side of Eq. (1), except that Kähler was already working with $v = \eta\overline{u}$, as proceeds for the electromagnetic coupling. That is next.

6 Dirac's equation for the electron

In Kähler's theory, the electromagnetic $4-$*potential* is represented by the difform

$$\omega = A_i dx^i - \Phi dt, \tag{33}$$

the meaning of A_i and Φ being obvious. The Dirac equation reads

$$i\hbar\partial u = -(iE_0 + e\omega) \vee u, \tag{34}$$

where E_0 and e are the mass and charge of the electron. Kähler then goes on to say that what is physically fundamental is not u but (literally)

$$|e| \cdot (u,\ \eta\overline{u}) = \rho w - (i_1 w_1 + i_2 w_2 + i_3 w_3), \tag{35}$$

having previously defined the w_i as $dx^k \wedge dx^l$, with (i,k,l) standing for $(1,2,3)$ and the other two even permutations. The four coefficients on the right of (35) are the coefficients of the $4-current$, which is a $3-form$. He then goes on to show that for any two solutions u and v of the Dirac equation, one has

$$d(u,\ \eta\overline{v})_1 = 0. \tag{36}$$

For electromagnetic coupling, implicit in (34) and now named as a, he has

$$\eta\overline{a} = a = \zeta a, \tag{37}$$

and

$$\partial(\eta\overline{v}) = -\eta\partial\overline{v} = -\eta(\overline{a} \vee \overline{v}) = -a \vee \eta\overline{v} = -\zeta a \vee \eta\overline{v}, \tag{38}$$

$\eta\overline{v}$ being a solution of the adjoint Dirac equation. Equations (38) and (39) are instrumental in showing that the conservation law

$$d(u,\ \eta\overline{u})_1 = 0 \tag{39}$$

follows for solutions of the Dirac equation with electromagnetic coupling.

We shall not enter into the minutiae of Kähler's discussion involving Eqs. (31)-(38), or of his previous treatment of the adjoint Dirac equation. This last equation is just a distraction and a source of misdirection for our future study of annulling the right-hand side of G1. In Eq (38), Kähler was implying (without actually saying it) that the right-hand side of G1is zero for electromagnetic coupling and that, therefore, (39) follows. Equation (39) does not by itself reveal what this equation entails at a greater level of sophistication. It hides much of the information that (2) contains and that we have derived.

7 Some implications of this alternative to the Copenhagen interpretation

Kähler failed to state the punchline of his calculation, namely that the conservation law for electrodynamics in quantum mechanics takes the form of (2) and that, therefore, the wave function is the amplitude of some conserved quantity that comes with both signs, unlike probability. Hence, the Copenhagen interpretation has heurisitic value but is not a fundamental tenet of quantum mechanics.

There is no obvious identification of a continuous distribution of something at the quantum level that comes with both signs, except for charge. That is our interpretation of the wave function, **charge amplitude**. This raises other issues, some of which are:

(1) If the wave function has this new interpretation, why is total charge always an integer multiple of a basic quantity, e? In a paper in progress, this author has found that neutrinos are better understood if we view protons and neutrons as constituted by the so-called core and (two) meson clouds rather than the fractional charges assigned to u and d quarks.[3]

(2) This new meaning of the wave function and the lack of fundamental significance of the Copenhagen interpretation suggests that the electron in the H atom hugs the proton rather than being a crazy point particle jumping all over the place around the proton. Electrons going through slits in a screen hug the latter while they go through it. The shape and size of the slits determine how the screen lets itself be hugged.

(3) QM done through the KC shows that the point just made about the electron's behavior when going through slits is just a particular case of a more general effect. It consists in that the inner behavior of systems that we call particles is more related to "its environment" (at times a classical apparatus) than the interpretation in classical terms of the microscopic world would seem to indicate. An example is that the angular operator $\frac{\partial}{\partial \phi}$ (or $i\hbar\frac{\partial}{\partial \phi}$ if your approach to it is through $r\ x\ p$ and the correspondence principle) does not yield just orbital angular momentum but also spin (i.e. inner or intrinsic).

See [1] from formula (22.2) to (22.3).

(4) Electrons have a definite mass under usual conditions. And yet they can also have other values for their mass, say in superconductors. Thus, a specific value of mass is not part of a faithful characterization of an electron. The issue then arises of what part of the electron's hypothetical wave function remains unchanged in all the electron's states and and what parts do not. In other words, what represents the essence of an electron?

(5) Most of the QM results obtained by Kähler hinge on the spinorial decomposition of u based on time translation symmetry. In obtaining those results, Kähler has not assumed special relativity, but only that the metric of flat spacetime is $dx^2 + dy^2 + dz^2 - dt^2$.

As E. Cartan already pointed out 100 years ago [4], the metric does not contain all the geometric reality of a space. The connection is as much a part of the picture. Without its being wrong, the guidance that the

Lorentz transformations provide is a poor representation of more comprehensive Lorentzian structure. I shall stop at that so that it not be thought that I am contradicting core physics, as I should only be extending it.

We have raised several new issues. There will be more. Relevant here are the viewpoints of Dirac, Bohr, and Heisenberg transmitted to us at the end of a talk by Heisenberg:

"Dirac often liked to say–and I always felt that it was a slight criticism– he felt that one can only solve one difficulty at a time. This may be right, but it was not the way I looked at the problems. Then I remembered that Niels Bohr used to say 'if you have a correct statement, then the opposite of a correct statement is of course an incorrect statement, a strong statement. But when you have a deep truth, then the opposite of a deep truth may again be a deep truth.' ...Therefore I feel that it is perhaps not only a deep truth to say 'You can only solve one difficulty at a time,' but it may also be a deep truth to say 'You can never solve only one difficulty at a time, you have to solve always quite a lot of difficulties at the same time' and with this remark perhaps I should close my talk."

...And now this author closes this paper with the remark that we shall have to solve several issues at the same time.

References

[1] Kähler, E. (1962): Der innere Differentialkalkül, Rendiconti di Matematica **21**, 425-523.

[2] Cartan, É. J. (1922): Leçons sur les invariants integraux, Hermann, Paris.

[3] Marmier, Pierre, and Eric Sheldon. Physics of Nuclei and Particles. Vol. 1. San Diego: Academic Press, 1986. ISBN: 978124731028.

[4] Cartan, É. J. (1922): Sur les equations de structure des espaces généralisés et l'expression analytique du tenseur d'Einstein, Comptes Rendus