THE FOUNDATIONS OF QUANTUM MECHANICS AND THE EVOLUTION OF THE CALCULUS.

Jose G. Vargas *

Abstract

In 1960-62, E. Kähler enriched Cartan's exterior calculus, making it suitable to address the needs, not only of quantum mechanics, but also of general relativity. Using his calculus, Kähler produced a "Kähler-Dirac" (KD) equation with which he reproduced the fine structure of the hydrogen atom. In addition, he showed that his equation's positron solutions correspond to the same positive energy as electrons.

In this paper, we present some basic concepts of differential geometry from a Cartan-Kähler perspective in order to understand, for instance, why the components of Kähler's tensor-valued differential forms have three series of indices. After a summary of some highlights of his calculus, we demonstrate its power by developing for the electron's large components their standard Hamiltonian beyond the Pauli approximation, but without resort to Foldy-Wouthuysen transformations or ad hoc alternatives. The same Hamiltonian is also derived for the large components of positrons, the latter particles not being identified with small components. Hamiltonians in closed form (i.e. exact through a finite number of terms) are obtained for both large and small components. The emergence of negative energies for positrons in the Dirac theory is interpreted from the perspective of the KD equation.

*PST Associates, 48 Hamptonwood Way, Columbia, SC 29209.

1 Introduction

In the mid to late 1920's, a few very gifted scientists developed the present formalism of quantum mechanics, and of QED. The merits of such formalism not withstanding, it has to be seen as an ad hoc response to the needs of the physics of the time. Eventually the calculus of differential forms evolved to the point where it could address the needs of quantum physics. In the last sections of this paper, we shall show that such a calculus constitutes a much better way to compute in quantum mechanics, specifically in relativistic quantum mechanics. But it is not only a matter of easier computing, as will transpire from the first sections of the paper.

The first of the two most important highlights in the evolution of the calculus is constituted by Cartan's introduction in a paper on differential equations of his exterior calculus of differential forms [1]. It is the modern language of differential geometry, differential topology and other branches of mathematics. It is based on exterior algebra. For decades, this calculus remained basically ignored except in Cartan and Kähler's work. The latter used it in 1934 to generalize the former's theory of exterior differential systems [2]. In 1960, Kähler generalized the exterior calculus itself, by endowing differential forms with the richer "Clifford" structure [3],.[4], [5].

Kähler's sophisticated calculus revolves around a basic equation that parallels the equation $df/dx = gf$, and $df/dx = 0$ in particular. It is called

the Kähler-Dirac or (simply) Kähler equation, to which he still referred as Dirac's. Among the many differences between the Dirac and Kähler equations (both give the same fine structure constant for the hydrogen atom, [4], [5]), there is the replacement of spinors with inhomogeneous differential forms with complex coefficients, differential forms which span a 32-dimensional differentiable manifold. All these are very significant features, with ab-initio-important consequences for quantum mechanics and even the foundations of geometry. As we shall show, there are several tantalizing (certainly intriguing) features in his calculus, which Kähler already showed and which we report in section 3.

His work, written in German and not yet translated, constitutes a formidable piece of mathematics. His theory is relativistic ab initio, but one does not need to know relativity to be able to use it. In fact, dealing with the issues that are normally associated with relativistic quantum mechanics (exception made of solving the hydrogen atom) requires just the simple version of this calculus that results when the use of Cartesian coordinates suffices. Being equation-based, a prior theory of physical observables is not needed, at least for the purposes that occupied Kähler and that will occupy us in this paper. Absent are also the gamma matrices.

In addition to the language barrier, an understanding of the Kähler calculus encounters the further barrier that one should be familiar with Cartan's

methods, which Kähler largely follows. A related complication is that the general quantities in his calculus have components with one series of superscripts and two series of subscripts. It is then clear that the widely held view that the calculus of differential forms pertains to antisymmetric covariant tensors clashes with the realities of this calculus, as it would appear that there are two types of antisymmetric covariant tensors. For this reason, we shall present in section 2 the Cartan-Kähler approach to vector fields, tensor-valued differential forms, and exterior, interior, covariant and Lie derivatives from. Fortunately, quantities with just one series of indices are sufficient to deal with problems such as the fine structure of the hydrogen atom and everything that we shall do in this paper.

In section 3, we summarize several outstanding features of his calculus and his equation, features which he himself discussed. In section 4, we briefly present further developments of this calculus, like removing the constraint of Levi-Civita affine connection of the space, explaining some of its ad hoc features and the rationale for some of its ansatzs. In the same section, we also raise the issue of the two "Dirac" equations that Kähler proposed, respectively in 1960 and 1962, for tensor-valued input differential form. Except for some minor reporting, the whole paper will be confined to scalar-valued input, whose components have just one series of indices. Kähler did not study any particular problem where the input required the use of more than

one series of indices.

In section 5, we discuss rotations, angular momentum, spinors and plane waves, all of it in the context of the two important concepts of Killing symmetries and constant differentials.

Readers thoroughly familiar with this calculus could just go directly to sections 6 to 8. They might even try to solve themselves the "graduate problems" that we proceed to formulate. The main contribution in this paper to the use of the Kähler calculus in physics is precisely the formulation of those problems, which are not difficult to solve, and which show the enormous power to do what the Dirac theory can only do with hole theory plus Foldy-Wouthuysen transformations, or similar paraphernalia.

Problem 1: At low energy, we write the wave form u of an electron in a slowly constant electromagnetic field as $e^{-imc^2t/\hbar}R(x,t) \vee \epsilon^-$, where m is the mass of the electron and ϵ^- suppresses positrons (i.e. projects on electrons). Find the system of equations satisfied by the even (φ) and odd (χ) parts of R.

Problem 2: Let us define $\chi_1 \equiv -\frac{i}{2m}P \vee \varphi$, where P denotes $dx^j \vee (-i\partial_j - eA_j)$ as P. Replacing χ with χ_1 in the system of problem 1, obtain the Pauli equation for φ.

Problem 3: Show that, in the next order of approximation for χ, one obtains in essence the same development of the Hamiltonian for φ as, say, in

Bjorken and Drell [8].

Problem 4: With positrons given in the form $e^{-imc^2t/\hbar}S(x,t)\vee\epsilon^+$, develop the Hamiltonian for the large components (now even) of S.

Problem 5: The wave form u of an electron in a constant electromagnetic field may also be written as $u = e^{-iEc^2t/\hbar}\Re(x)\vee\epsilon^-$, where E is the energy of the system. Equate the R and $\Re$ forms of u to obtain simple expressions for $\varphi_{,t}$ (also $\chi_{,t}$) in terms of φ (respectively χ). Use that to readily obtain an exact Hamiltonian for χ containing just a few terms.

Problem 6: Transform the coupled system of equations for φ and χ so that their roles in that system will be exchanged. Relate to the negative energy solutions of the Dirac theory the term that jumps equation in that transformation.

Problem 7: Based on the treatment of problems 1 and 2, develop the iterative procedure that suggests itself to obtain χ,and then the Hamiltonian for φ, in ever increasing orders.

Problem 8: Obtain an exact (i.e. in closed form) Hamiltonian for φ using the idea given in problem 5. Compare the result with the also exact Hamiltonian for χ previously obtained.

A point about terminology follows. There are subtleties familiar to algebraists (a) between the concepts that go by the terms of exterior and Grassmann algebras, and more so (b) with the use of the terms Grassmann

algebra, Clifford algebra and Clifford-Atiyah algebra. To follow Graff [6], in (a), the difference is that, whereas exterior algebra is based just on the exterior product, Grassman algebra contains the additional structure conferred by an inner product induced by a quadratic form. Clifford algebra contains only the Clifford product (example, the algebra of Dirac gamma matrices). The Kähler-Atiyah contains both the Grassmann and Clifford algebras as substructures [6]. However, the Clifford product of two vectors (similarly for vector and multivector, but not for two general multivectors) can be decomposed into their symmetric and antisymmetric parts, identifiable with the exterior and interior products in this case. There is not, therefore, much point for physicists in making such subtle differences as to what constitutes Clifford algebra. Hence, we shall adopt the practical perspective, common in the literature, of using the term Grassmann algebra for exterior algebra and the term Clifford algebra for the Kähler-Atiyah algebra. The term Kähler algebra will be reserved for the so understood Clifford algebra (i.e. Kähler-Atiyah algebra) of cochains, i.e. of functions of hypersurfaces. As we shall explain later in the paper, we shall use the term differential forms for these cochains, in accordance with Kähler, and because the cochains rather than the antisymmetric multilinear functions of vectors are of the essence of his calculus [7].

2 The Cartan-Kähler View of Basic Concepts of Geometry and the Calculus

A comprehensive treatment of tangent vectors is given in an authoritative book by Choquet-Bruhat et al. [9]. They define them (a) as *sets of quantities* that transform in a particular way, (b) as *linear operators* on spaces of functions (active vectors) and (c) as *equivalence classes of curves* (passive vectors). Defining concepts by their transformation properties is clearly inadequate. For instance, a 1-cochain, i.e. a function of curves, and a field of linear functions of vectors transform in the same way. Similarly, a tangent vector field referred to the reciprocal basis $\mathbf{e}^i$ (defined by $\mathbf{e}^i = g^{ij}\mathbf{e}_j$) has components which transform like 1-cochains and fields of linear functions of vectors. Based just on transformation properties, there would be no point in distinguishing between the two series of subscripts in the Kähler calculus

The vectors defined in (b) are operators consisting of linear combinations of the partial derivatives with respect to the coordinates. Cartan refers to them as infinitesimal transformations [10]. Both he and Kähler do not refer to those operators as vectors, theirs being of type (c), as they do not act on anything. In other words, they do not play the active role that the infinitesimal operators play. When Cartan generates a differential $(r-1)$-form from a pair constituted by a differential r-form together with an infinitesimal operator, he is doing what in the modern language is called the evaluation

of a differential form on a vector field [10]. But he does not refer to the infinitesimal operator as a vector field. Nor does he so when he makes that operator act on the differential r-form to obtain another differential r-form (operation nowadays called the Lie derivative of a differential form).

The point just made is very clear in Cartan's approach to differential geometry. He starts [11] with the concept of what nowadays is known as an elementary or Klein geometry [12], [13], [14], [15]. These are flat spaces, but are not viewed as just the easiest examples of general spaces (affine, projective, conformal, etc.). Retrospectively, the Klein geometries certainly fit this mold of being the easiest examples, but they constitute the primary concept. Given a "flat space" of a given type, one differentiates twice the group equations defining it. These are the equations of structure, which, in this case make the connection integrable ab initio. The generalized case corresponds to when equations of structure of the same form but with different input no longer amount to integrability conditions (in other words, when these conditions are broken) [14].

We proceed to explain the approaches to differentiation by Cartan and Kähler. The former uses exterior derivatives. Kähler uses more general differentiations, containing interior and exterior parts. The latter part constitutes Cartan's exterior derivative, or simply Cartan's derivative. It gives the exterior derivative when acting on scalar-valued differential forms, and the co-

variant derivative on tensor-valued differential $0-$forms , i.e. on tensor fields (His rather casual approach to differentiation has been formalized by Flanders [16]). Those authors use the term exterior even when d acts on vector-valued forms, dd then not yielding zero in general (see Eq. (3)). Kähler's combined exterior-interior derivative (which he called interior derivative) does not satisfy the Leibniz rule, not even when applied to scalar-valued differential forms. Those operations can still be viewed as derivatives from some more general perspective.

We return to vector fields. They are passive in books on the theory of curves and surfaces in Euclidean space (typical representatives being Struik [17] and Henri Cartan [18]), or in books like Elements of the Tensor Calculus by Lichnerowicz [19]. Of course, one can do flat space geometry with active vector fields, but it is not intuitive and runs counter to what one learned in, say, courses on the multivariable calculus. The key point here is that the passive concept of vector field in generalized spaces as equivalence classes of curves bodes well with the simple concept of vector field that is standard in the study of Euclidean space. In both cases we are dealing with passive vector fields, however defined.

Exterior differentiation of $\mathbf{v}$ in the extended sense of Cartan, Kähler and Flanders yields

$$d\mathbf{v} = d(v^i \mathbf{e}_i) = dv^i \mathbf{e}_i + v^i d\mathbf{e}_i = dv^i \mathbf{e}_i + v^i \omega_i^j \mathbf{e}_j. \tag{1}$$

Differentiating again, one gets

$$dd\mathbf{v} = (0 - dv^i \wedge \omega_i^{\cdot j}\mathbf{e}_j) + (dv^i \wedge \omega_i^{\cdot j}\mathbf{e}_j + v^i d\omega_i^{\cdot k}\mathbf{e}_k - v^i\omega_i^{\cdot j} \wedge \omega_j^{\cdot k}\mathbf{e}_k). \qquad (2)$$

We have used parentheses to make clear the origin of each of the five terms on th eright of Eq. (2), respectively from $dv^i\mathbf{e}_i$ and $v^i\omega_i^j\mathbf{e}_j$. We further write

$$dd\mathbf{v} = v^i(d\omega_i^{\cdot j} - \omega_i^{\cdot j} \wedge \omega_j^{\cdot k})\mathbf{e}_k = v^i\Omega_i^{\cdot k}\mathbf{e}_k, \qquad (3)$$

where the $\Omega_i^{\cdot k}$ constitute the differential forms.defining the affine curvature

Next, consider the concept itself of differential forms. Modernly, these are defined as fields of antisymmetric multilinear *functions of vectors*, though, in the same breath, some authors say that they are *integrands*. However, an integrand is above all a function of hypersurfaces, whose evaluation is its integration. It is in this way that Rudin [20] defines differential forms, which is consistent with the Kähler calculus, where, as in the Cartan calculus, the same operator yields different derivatives depending on the mathematical object it acts upon.

As we said, the components of Kähler's tensor-valued differential forms have one series of superscripts and two series of subscripts [3], [5]. The general formula that he gives for the derivatives speaks of the nature of the object to which each series of subscripts refers. The exterior part d of this derivative is the usual covariant one if the second series of subscripts is empty. But d is the exterior derivative if the series of superscripts and first series of subscripts

are empty. It is clear that, since the action of d on tangent tensors (in the sense of Cartan, Kähler and Flanders) depends on connection, it must also depend on connection for multilinear functions of tangent vectors, whether antisymmetric or not. On the other hand, functions of hypersurfaces do not depend on connection, since they live on the base manifold, not on its tangent bundles. Hence, the differential forms of Kähler, like those of Cartan and Rudin, are essentially cochains, not antisymmetric multilinear functions of vectors.

Consider finally Lie derivatives. In a chapter titled "The differential systems that admit an infinitesimal transformation" [10], Cartan extends the concept of such transformations from the ring of functions to the ring of differential forms. The extension is a matter of straightforward but cumbersome computations once the equation $Xdx^i = dXx^i$ has been reached, X being the infinitesimal transformation. Cartan does not perform such direct computations or produce a formula for the explicit purpose of avoiding them. He, however, proceeds to prove a theorem relating the action of the infinitesimal operator on the form to the action of the form on the operator. The latter action is very easy to compute; the theorem can then be used to easily obtain the first action, i.e. what is nowadays known as the Lie derivative of the form. Hence, he produces after all a formula which can be used to easily compute the Lie derivative of differential forms, even if it comes in the form

of a theorem with a different purpose.

When referring to the work by Cartan just mentioned, Kähler proceeds to use the term Lie derivative instead of infinitesimal transformation. He reaches a formula for computing it [3]. It does not take quite the same form as the equation that states the foregoing Cartan theorem. Kähler first pulls the differential form to a new coordinate system y. Define α^i in the given coordinate system x through $X \equiv \alpha^i \partial/\partial x^i$ and construct the differential system

$$\frac{dx^i}{d\lambda} = \alpha^i(x^1, ..., x^n), \qquad\qquad i = 1, ..., n. \tag{4}$$

The coordinate y^n is λ, and the y^i $(i = 1, ..., n-1)$ are $n-1$ independent first integrals, independent of the constant which is additive to λ. The pull-back to the y coordinate system of a differential p-form

$$u = \frac{1}{p!}\, a_{i_1 ... i_p}\, dx^{i_1} \wedge ... \wedge dx^{i_p} \tag{5}$$

is

$$u = \frac{1}{p!}\, a_{i_1 ... i_p}\, \frac{\partial x^{i_1}}{\partial y^{k_1}} ... \frac{\partial x^{i_p}}{\partial y^{k_p}}\, dy^{k_1} \wedge ... \wedge dy^{k_p}. \tag{6}$$

Kähler shows that acting on (2.6) with $\frac{\partial}{\partial y^n}$ one gets

$$\frac{\partial u}{\partial y^n} = \frac{1}{p!}\, (X a_{i_1 ... i_p})\, dx^{i_1} \wedge ... \wedge dx^{i_p} \;+\; d(\alpha^i) \wedge e_i u, \tag{7}$$

where $X a_{i_1 ... i_p}$ is, of course, $\alpha^j \partial_j a_{i_1 ... i_p}$. We have copied this formula verbatim except for the parenthesis around $X a_{i_1 ... i_p}$, but it should rather be written

with a pull-back symbol on the left (for details see [15]). The operator e_i is a contraction: for each term in u, one places the factor dx^i at the front of each string of products of differential 1-forms, with a change of sign if applicable, and eliminates that factor. This action is zero on terms where dx^i is not present [3], [5].

Iin 1962, Kähler defined the Lie derivative of a tensor-valued differential form. He did so seeking (verbatim) that

$$(\sigma X u)^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda} = \frac{\partial}{\partial y^n} (\sigma u)^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda}, \tag{8}$$

where σ is the pull-back operator to the y coordinate system (symbol which he uses instead of σ^*) and where the k and i indices are valuedness indices. Each of the $(\sigma u)^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda}$ is a differential form; hence a third series of indices is implicit in it. Kähler does not resort to any commutator of his operator X with the tangent and cotangent (valuedness) fields. He proves that, in terms of arbitrary coordinate systems, one has:

$$\begin{aligned} (Xu)^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda} &= \alpha^i \frac{\partial}{\partial x^i} u^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda} + d(\alpha^i) \wedge e_i u^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda} \\ &- \frac{\partial \alpha^{k_1}}{\partial x^r} u^{r \ldots k_\mu}_{i_1 \ldots i_\lambda} - \ldots - \frac{\partial \alpha^{k_\mu}}{\partial x^r} u^{k_1 \ldots r}_{i_1 \ldots i_\lambda} \\ &+ \frac{\partial \alpha^r}{\partial x^{i_1}} u^{k_1 \ldots k_\mu}_{r \ldots i_\lambda} + \ldots + \frac{\partial \alpha^r}{\partial x^{i_\lambda}} u^{k_1 \ldots k_\mu}_{i_1 \ldots r}. \end{aligned} \tag{9}$$

Commutation relations still emerge in any case, as shown by the fact that his formula (9) becomes

$$Xu^k = \alpha^i \frac{\partial u^k}{\partial x^i} - u^i \frac{\partial \alpha^k}{\partial x^i} \tag{10}$$

for a vector field u^k. Thus formula (9) is consistent with the modern concept of Lie derivative of a vector field as the commutator,

$$\mathfrak{L}_X Y = [X, Y] \equiv XY - YX. \tag{11}$$

In spite of the correspondence between equations (10) and (11), the right hand side of equation (8) may give the a first impression that something is wrong with it, given the absence of the commutator. This absence reflects that the $-YX$ term is zero in the y coordinate system, where the components of X are $(0, 0, ..., 1)$. Finally, as previously reported, Cartan did not deal with Lie derivatives (i.e. infinitesimal operators) of vector fields, and one wonders what Kähler would need them for, since he does not use them.

It is worth recalling in connection with equation (9) the important point made before to the effect that the symbol for differential form represents cochains in Kähler's work. In this regard it is very instructive to consider Slebodzinski's treatment of the subject [21]. Let v be a vector field. He obtains Xv using that the contraction of u and the differential 1-form β is a scalar. He uses for that the Lie derivative of β and the Leibniz rule. However, in first obtaining the formula for the Lie derivative of differential forms, Slebodzinski treats them as integrands, i.e. as functions of hypersurfaces, whose d derivative in the sense of Kähler does not depend on connection. He does not consider them as fields of multilinear functions of vectors, whose d derivative, like that of vectors fields, depends on connection. Thus, he

actually contracts the vector field with a cochain, not with a linear function of vector fields.

Our observations with regards to Lie derivatives of vector fields point to the fact that there is nothing in them of geometric significance, even though they are used in the most common modern definition of the affine curvature and torsion, from which one then obtains the equations of structure (This approach need not be the best, since one can obtain those equations in a much simpler yet rigorous way and fully in the spirit of Cartan) [22]. The lack of significance has to do with the fact that equality of two tangent vectors at two different points of a differentiable manifold (and, therefore, any significant derivative of vector fields and their duals) is determined by the affine connection, and we can define any we want. Sure enough one could have two different concepts of derivatives of vector fields, like one has the concept of exterior and Lie derivatives of scalar-valued differential forms. The question arises, however, of what is the geometric value of having a partial differentiation (see Eq. (8)) of a tensor-valued differential form (i.e. the limit of a quotient of increments) if the increment in the numerator lacks meaning because the tangent spaces at the points x and $x+dx$ are not comparable. For comparison one needs a connection. No connection enters the Lie derivative equations, (8)-(11). The existence of symmetries is irrelevant in this respect, since one can choose any connection one wants regardless of what symmetries

one has. A truly geometrically significant Lie derivative of vector fields would have to involve the connection. Of course, there are many applications where the connection is zero in some particular frame field; ignoring the connection in such a field when it should not be ignored then amounts to making it zero and thus to postulating teleparallelism (TP), but not necessarily global, i.e. strict TP.

3 Quantum Mechanical Issues Raised by Kähler's Own Work

The combination of Dirac's authoritative formalization of the principles of quantum mechanics [23] and the Copenhagen interpretation did not constitute a solid enough foundation to preempt what Mead [24] (page 5) refers to as the "muddle and fuss over theory" in the decades that followed. With regards to the Dirac equation in particular, Thaller [25] had this to say in the preface of his book on this subject:

> "Perhaps one reason that there are comparatively few books on the Dirac equation is the lack of an unambiguous quantum mechanical interpretation. Dirac's electron theory seems to remain a theory with no clearly defined range of validity, with peculiarities at its limits which are not completely understood."

All this is not surprising since the development of those principles and interpretation took place to accommodate the new experimental evidence about matter through the adoption of new versions of concepts inherited from a tradition where matter is particles to the exclusion of any wave concept. The need for new versions of those concepts rather than altogether new concepts may be unavoidable in any useful quantum physics. However, more refined perspectives on their use may arise if those concepts are not introduced like with forceps, as happened in the 1920's, in order to reach evolution equations of observables, and wave equations. In contrast, the Kähler equation requires none of this. It emerges on its own right in his calculus. The different placement of the unit imaginary in the Kähler and Dirac equations should be more than enough illustration of this point. Following Kähler, we shall not presuppose interpretations. We shall rather let the mathematics speak, like he did in the matter of spin and total angular momentum, and of particles and antiparticles, with striking consequences in both cases. But we shall not try to rush responses to issues that the present development of the calculus does not yet wish to speak of.

We proceed to report on some highlights of this calculus that are (almost0 explicit and not emphasized in Kähler's papers of 1960 and 1962. From now on, and because of the physical applications, we shall use Latin indices for 3-space, and Greek indices otherwise.

(a) The Kähler equation is

$$\partial u = a \vee u, \tag{12}$$

where $\vee$ stands for Clifford product of differential forms, a is the input differential form and u is the output differential form. In this paper, ∂ designates Kähler's general derivative operator (not partial differentiation unless used as in $\partial/\partial u^k$). It constitutes what we shall refer to as the sum of the exterior and interior derivatives (or exterior and interior covariant derivatives when ∂ acts on tensor-valued differential forms):

$$\partial u = du + \delta u, \tag{13}$$

where

$$du = dx^\mu \wedge d_\mu u \tag{14}$$

and

$$\delta u = dx^\mu \cdot d_\mu u. \tag{15}$$

and where "$\cdot$" stands for interior product. Kähler proves that, for the Levi-Civita connection, δu becomes $^*d^*$ or -$^*d^*$, the sign depending on signature and dimensionality of the space. For present purposes, we do not need the details of what goes into the making of the operator d_μ, which Kähler calls the covariant derivative. Suffice to say that it allows one to view the exterior derivative d and (in the case of the Levi-Civita connection) the coderivative δ

in the new light of equations (14) and (15). The operators d and d_μ satisfy the Leibniz rule, but ∂ and δ do not. Equations (13)-(15) should be motivation enough to still use the term derivative in those cases, disregarding the modern use of the term.

It is to be noted that, if the exterior-interior calculus of differential forms had been formulated before the advent of quantum mechanics, the natural order of differential equations to be solved that involve ∂ might have been as follows: $\partial u = 0$ (definition of strict harmonic differential), $\partial\partial u = 0$ (definition of harmonic differential), $\partial u = a$ (of this type is the Maxwell system, with a of grade 3 and u of grade 2) and $\partial u = a \vee u$. In other words, the basic equation of quantum mechanics would have emerged in the calculus without input from the physics, had humans had greater mathematical ability.

Let e_μ be the operator as in Eq. (8) and define e^μ as $g^{\mu\nu}e_\nu$. The operator ∂ satisfies [5]

$$\partial\partial u = g^{\mu\nu}d_\mu d_\nu u + R_{\mu\nu}dx^\mu \vee e^\nu u - \Omega_{\mu\nu} \vee e^\mu e^\nu u, \tag{16}$$

regardless of whether u is scalar-valued or not. $R_{\mu\nu}$ stands for the components of the Ricci tensor and $\Omega_{\mu\nu}$ stands for the curvature 2-forms This suggests that the generalization to curved spacetime of the Laplacian of a differential form should not be viewed as just the generally covariant form $g^{\mu\nu}d_\mu d_\nu$, unless

the curvature is zero. Using here the Kähler equation, we have

$$\partial(a \vee u) = g^{\mu\nu} d_\mu d_\nu u + R_{\mu\nu} dx^\mu \vee e^\nu u - \Omega_{\mu\nu} \vee e^\mu e^\nu u \tag{17}$$

which one would further develop on the left, showing the entanglement of the curvature with the Kähler equation. Of special interest would be the case when a and u were geometric quantities themselves, but this exceeds the scope of this paper.

To summarize, Kähler's common mathematical formalism for QM and general relativity is constituted by his generalization of Cartan's calculus of differential forms, and may actually entangle those branches of physics, as intimated by the combination of the pair of equations (12) and (16). This entanglement, if and when it is physically found, will have to be interpreted as the need to represent quantum mechanical states by differential forms rather than by functions (or even by spinors, under the old concept of the Laplacian operator).

(b) The Kähler equation has a conservation law built into it. Let r and s be any two differentiable differential forms. He defined

$$\langle r, s\rangle \equiv (\zeta s \vee r)_0 \, z, \qquad \langle r, s\rangle_1 \equiv (\zeta s \vee dx^\mu \vee r)_0 \;\; e_\mu z, \tag{18}$$

where $(\ \_\ )_0$ denotes the 0-form part of the contents of the parenthesis, where ζ denotes reversion of the order of all the $1-$form factors in the differential forms, and where z is the unit differential $n-$form. $\langle r, s\rangle$ and $\langle r, s\rangle_1$ are

respective $n-$forms and $(n-1)-$forms. For later use in subsection (f), it is worth pointing out that

$$\langle r, s\rangle = \sum_{m=0}^{n} \langle r_m, s_m\rangle \,, \tag{19}$$

and

$$\langle r_m, s_m\rangle = \left(\sum_{i_1<\ldots<i_m} e_{\mu_1}\ldots e_{\mu_m} r_m \wedge e^{\mu_1}\ldots e^{\mu_m} s_m \right) \; z, \tag{20}$$

where r_m and s_m denote the terms of degree m in the generally inhomogeneous differential forms r and s respectively. The last two equations together imply that, if the metric is positive definite, the coefficient of z in the corresponding expression for $\langle r, r\rangle$ is positive definite if r is not null.

A generalized Green's formula of this calculus reads:

$$\langle r, \partial s\rangle + \langle s, \partial r\rangle = d\,\langle r, s\rangle_1 \,. \tag{21}$$

The Kähler equation admits a conjugate equation

$$\partial v = -\zeta a \vee v \tag{22}$$

whose relevance lies in that the sum $\langle u, \partial v\rangle + \langle v, \partial u\rangle$ is zero if u and v are respective solutions of a Kähler equation and its conjugate. The conservation law

$$d\,\langle u, v\rangle_1 = 0, \tag{23}$$

thus follows. Notice that the unit imaginary does not enter here at this point.

To summarize: in principle, the existence of a conservation law associated with the main equation of relativistic quantum mechanics does not require a complex algebra.

(c) In 1961 and again in 1962, Kähler solved the fine structure of the hydrogen atom with the specific equation

$$\partial u = -\frac{1}{\hbar c}(mc^2 + ice\phi dt) \vee u \tag{24}$$

and signature +2. Here ϕ denotes the Coulomb potential and m is the mass of the electron. Notice the mismatch with the Dirac theory in what concerns the non-standard position of the unit imaginary.

The relation between the solutions to the fine structure of the hydrogen atom through the Dirac and Kähler equations emerges as one exploits the time translation and cylindrical rotational symmetries in the latter of the two equations. Four different mutually annulling primitive idempotents arise as factors in respective four terms into which the general solution decomposes (details provided under (e)). They make the difference between those individual solutions and the solution of the Dirac equation, which lacks a similar structure, and to which they are otherwise equivalent. The larger spherical symmetry does not give rise to additional primitive idempotents and thus to further decomposition of the general Kähler solution.

To summarize, the Kähler equation has a much richer structure of solutions than the Dirac equation. This coexistence of equations shows that

the location of the unit imaginary is formalism dependent. The argument that leads to a unit imaginary in the four momentum operator –and thus on the left hand side of the Dirac equation– was based on the search for a Hamiltonian written in terms of quantum mechanical operators that parallels corresponding classical mechanical operators. That argument is, therefore, begging for an explanation that might one day come from further development of Kähler's theory.

(d) Equation (7) does not involve metric structure, much less symmetries of the metric. The latter give rise to Killing equations, $d_\mu\alpha^\nu + d_\nu\alpha^\mu = 0$, which allow one in turn to further write the rotations' Lie operator on differential forms, (7), as [5]

$$Xu = \alpha^\mu d_\mu u + \frac{1}{4} d\alpha \vee u - \frac{1}{4} u \vee d\alpha. \tag{25}$$

where α is defined by

$$\alpha = g_{\mu\nu}\alpha^\mu dx^\nu. \tag{26}$$

When this formula is applied to the rotation operator around the axis x^i, one gets [5]

$$X_i u = x^k \frac{\partial u}{\partial x^l} - x^l \frac{\partial u}{\partial x^k} + \frac{1}{2} w_i \vee u - \frac{1}{2} u \vee w_i, \qquad i = 1, 2, 3 \tag{27}$$

where

$$w_i \equiv dx^k \wedge dx^l = dx^i \vee w = w \vee dx^i, \tag{28}$$

with (i, k, l) a cyclic permutation of (1,2,3) in (27) and (28), and with w defined as $dx^1 \wedge dx^2 \wedge dx^3$ (i.e. the z of three dimensions). The action of the spin operator is to be read from the last two terms of Eq. (27), which we use to summarize this subsection with Kähler's statement that "the spin of the electron is interpreted as the need to represent the state of the electron by a differential form rather than by a function".

(e) The considerations in the present and next subsections are related to the particle/antiparticle issue. Time-translation symmetry is of the essence of the argument. It is important, however, to keep track of the point at which that symmetry plays a role.

Following Kähler, any spacetime differential can be written as $u = u' + u'' \wedge dt$, where u' and u'' are uniquely defined by the condition that they be *spatial differentials*, i.e. that they do not contain dt as a factor. Define then the mutually annulling constant differential idempotents

$$\epsilon^{\pm} \equiv \frac{1}{2} \mp \frac{ic}{2} dt \tag{29}$$

(a constant differential u is defined by the condition $d_\mu u = 0$). The i factor would not have been necessary to make a dt related idempotent if the signature had been -2. We wish, however, to follow Kähler in using +2, since we are reproducing in more targeted form some of the implications of his

computations. One readily finds that

$$\epsilon^+ + \epsilon^- = 1, \qquad dt = i\frac{\epsilon^+ - \epsilon^-}{c}. \tag{30}$$

Substitution of these equations in $u = u' + u'' \wedge dt = u' \vee 1 + u'' \vee dt$ yields an expression of the form

$$u =^+ u \vee \epsilon^+ +^- u \vee \epsilon^-, \tag{31}$$

where ${}^+u$ and ${}^-u$ are uniquely defined by the process just stated. Although both ${}^+u$ and ${}^-u$ are different from u, one still has

$$u \vee \epsilon^\pm =^\pm u \vee \epsilon^\pm. \tag{32}$$

as follows from multiplication of Eq. (31) by $\epsilon^\pm$ on the right, and taking into account that ϵ^+ and ϵ^- are mutually annulling idempotents.

Constant differentials, c, satisfy the property that, if u is a solution of a Kähler equation, $u \vee c$ also is a solution of the same equation.. Thus the $u \vee \epsilon^\pm$ are solutions of the same equation as u, and so are then the ${}^\pm u \vee \epsilon^\pm$ by virtue of equations (32). Since

$$u \vee \epsilon^\pm \vee \epsilon^\pm = u \vee \epsilon^\pm, \qquad u \vee \epsilon^\pm \vee \epsilon^\mp = 0, \tag{33}$$

we may view $u \vee \epsilon^\pm$ as proper differential forms with proper value 1 (corresponding to right multiplication by $\epsilon^\pm$), and with proper value 0 (corresponding to right multiplication by $\epsilon^\mp$).

Up to this point no assumption has been made about the symmetries of the factor a in the Kähler equation. When a does not depend on time, the action on a differential form of the Lie operator corresponding to time translations is a constant times $\partial u/\partial t$. Without justification, Kähler selects the constant so that this operator is the standard energy operator $i\hbar(\partial u/\partial t)$ of the Dirac formalism. It is then clear that the solutions of the Kähler equation that are eigen solutions of this operator are of the form

$$u = p^+ \vee T^+ + p^- \vee T^-, \tag{34}$$

where

$$T^{\pm} = e^{-\frac{i}{\hbar}Et}\left(\frac{1}{2} \mp \frac{ic}{2}dt\right), \tag{35}$$

and where the spatial differentials $p^{\pm}$ do not depend on t either (thus called *pure spatial differentials*). Both solutions

$$^{+}u = p^+ \vee T^+, \qquad ^{-}u = p^- \vee T^-, \tag{36}$$

pertain to the same sign of the energy, which is chosen as positive.

We summarize this subsection with a partial paraphrasing of a Kähler statement on spin. Negative energy solutions and the inseparability of pure-particle and pure-antiparticle solutions for bound systems are spurious effects of representing the state of electrons and positrons by spinors rather than by differential forms.

(f) The argument outlined under (e) is not enough to justify the use of the term antiparticle. Kähler shows that, for minimal electromagnetic coupling, $\eta\bar{u}$ is a solution of the conjugate Kähler equation, where η is the operator that reverses the sign of each $1-$form factor in differential forms and where overbar stands for complex conjugate. The conservation law $d\langle u, v\rangle_1 = 0$ then includes $d\langle u, \eta\bar{u}\rangle_1 = 0$. He then shows that

$$[u, v] = [{}^{+}u \vee \epsilon^{+},\ {}^{+}v \vee \epsilon^{+}] + [{}^{-}u \vee \epsilon^{-},\ {}^{-}v \vee \epsilon^{-}], \tag{37}$$

where the notational simplification $[u, v] \equiv \langle u, \eta\bar{v}\rangle_1$ has been used. Spatial differentials $(p, q,...)$ satisfy that $[p, q] \equiv \{p, \eta\bar{q})_1 \wedge icdt$, where curly brackets play with respect to the spatial metric the same role as $\langle\ \ \rangle$ plays with regards to the spacetime metric. He finally shows that

$$[u,\, u] = \left[{}^{+}u,\ {}^{+}u\right] + \left[{}^{-}u,\ {}^{-}u\right], \tag{38}$$

where

$$\left[{}^{\pm}u,\ {}^{\pm}u\right] = \pm\frac{1}{2}\{{}^{\pm}u,\ {}^{\pm}u\} + \frac{1}{2}\{{}^{\pm}u,\ \eta^{\pm}u\}_1 \vee icdt. \tag{39}$$

The $\pm$ sign at the front of the first term on the right (while the second sign remains the same) is interpreted to mean that, for the same sign of the energy, the two solutions in ${}^{\pm}u$ are each other's antiparticle.

Equations (39) have to be compared with the general form for currents which, as explained in the previous paper, are given by

$$j = \rho w - (j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy) \wedge dt. \tag{40}$$

Hence, following Kähler, the 4-current has to be identified with $|e| \langle u, \eta\bar{u} \rangle_1$. Thus, for the electron, we have:

$$\rho w = -\frac{|e|}{2}\{^{-}u,^{-}u\}, \tag{41}$$

and, since $\{^{-}u,^{-}u\}$ is proportional to w, and $w^2 = 1$, the density is given by

$$\rho = \frac{|e|}{2}\{^{-}u,^{-}u\}w, \tag{42}$$

where the symbol of Clifford product by w is not required, for obvious reasons.

In the same way as we have read a density, we can read a corresponding 3−current for the electron (really a 3−form) by identifying the 3-current part of (40) with $|e|$ times the 3-current part of (39). For the electron, we have,

$$-j_i dx^k \wedge dx^l \wedge dt = \frac{|e|}{2}\{^{-}u, \eta^{-}u\}_1 \vee icdt, \tag{43}$$

where the presence of the ordered triple (i, k, l) means sum over cyclic permutations of $(1, 2, 3)$ (and similarly $\rho w = \frac{|e|}{2}\{^{+}u,^{+}u\}$ for the positron).

4 Further Developments of the Kähler Calculus

The success with the fine structure constant of the hydrogen atom and the just reported results by Kähler speak of the potential significance for physics of his calculus. Before we discuss it further and in order to provide perspective, let us consider recent extensions of the Kähler calculus.

In subsections (a) to (b), we deal with some unexplained issues in Kähler's work on this subject, namely (a) the three series of indices in components of his tensor-valued differential forms, (b) his ansatz for differentiation and (c) the dependence of the interior derivative on connection. In subsections (d) and (e), we deal with the replacement of tensor-valuedness with Clifford-valuedness of his differential forms, thus incorporating the product $\mathbf{e}_i.\mathbf{e}_j$, which is one of the cornerstones of Cartan's theory of moving frames. In subsection (f), we argue that the results obtained in (d) and (e) address an issue raised in Cartan's writing on geometry and that he left unresolved. We also provide a geometric context to a Cartan innovative way of writing the equations of the physics in a five dimensional space.

(a) Kähler's tensor-valued differential forms have three series of indices, one of them being of superscripts and the other two of subscripts. This difference with everybody else's differential forms has to be inferred from some of his main formulas, since he gives no reason for it. For an explanation as to what those three series of indices refer to see [7]. Readers can also find there the justification of why his differential expressions do not mean the usual elements of the exterior cotangent algebra, i.e. antisymmetric covariant tensors, but rather functions of hypersurfaces. One can imagine that Kähler's silence on this intended to avoid polemics.

(b) Kähler produces cumbersome ad hoc expressions for what he calls

the covariant derivative, $d_\mu u$. As reported in section 3, he contracts the latter's exterior and interior products with dx^μ to obtain the exterior covariant and interior covariant derivatives respectively (In the following, these derivatives will be called simply exterior and interior). Since these products come together into the Clifford product, the corresponding derivatives also come together into what we shall call Kähler derivative, to which he refers as interior. Those intimate relationships may have constituted his justification for the use of the term derivative, disregarding whether the Leibniz rule is satisfied or not. In addition to their ad hoc character, the lengthy expressions proposed by Kähler apply only to manifolds endowed with the Levi-Civita connection. The following alternative course of action is possible.

The exterior product of two unknown differential forms does not imply what their interior product is. Hence, the exterior derivative does not imply what the interior derivative is. However, once a connection is given, one can use covariance constraints to determine canonical covariant and interior derivatives, thus removing the aforementioned ad hoc character. This can be done with arbitrary affine connection through the first equation of structure [26], [27].

(c) The interior derivative of even scalar-valued functions of hypersurfaces depends on connection (In the case of Levi-Civita's connection, this is disguised by the fact that the interior derivative becomes the coderivative,

a prior concept introduced directly through the metric). We have shown that said dependence is a concomitant of requiring that the equations of the derivatives of the connection will associate bases of 1-cochains with bases of tangent vectors [27].

(d) Though the Kähler equation of 1960 is well defined also when the input differential form a is tensor valued, it has the unpleasant feature of run-away valuedness, arising from the fact that $a \vee \_$ is a degree-changing operator and ∂ is not [7]. Without explanation but likely because of that feature, Kähler produced in 1962 a different equation for tensor-valued "a" (For a detailed discussion see [28]). Unaware of his 1962 equation, the same situation was addressed [29] $a$in the alternative way that we proceed to summarize. When Kähler writes $u \vee v$ for two tensor-valued differential forms, he actually means $u(\vee, \otimes)v$, where $\vee$ takes place in the algebra of differential forms and $\otimes$ takes place in the algebra of valuedness tensors. For different reasons, we proposed to replace the tensor-valuedness with Clifford-valuedness. A "new Kähler equation", $\partial u = a(\vee, \vee)u$, results [29]. The tangent Clifford product includes the product $\mathbf{e}_\mu.\mathbf{e}_\nu\,(= g_{\mu\nu})$, which gives rise through differentiation to the relation of compatibility of metric and affine connection.

(e) The tangent Clifford algebra resolves the problem of run-away valuedness but creates a new one. Consider the translation form $d\mathbf{P}$ of the theory of moving frames. It is a vector-valued differential 1-form. The prod-

uct $d\mathbf{P}(\vee,\vee)d\mathbf{P}$ yields $dx^\mu \wedge dx^\nu \mathbf{e}_\mu.\mathbf{e}_\nu + 4$. This last term lacks geometric interpretation other than the trivial one, the dimension of the space. For this reason, among other, a double Clifford algebra was not introduced in spacetime itself, but rather in a Kaluza-Klein type of space suggested by Finslerian TP [29], where the horizontal differential invariants are (ω^μ, ω^i_0) [22]. In appropriate frame fields of the Finsler bundle, the ω^i_j (and not just their pull-back to the fibers) become the left-invariant forms of the rotation group in three dimensions. $d\wp$ ($\equiv d\mathbf{P} + \mathbf{u}d\tau$) is the translation element in that 5-dimensional space, τ being proper time, $\mathbf{u}$ being its dual tangent vector field and $d\mathbf{P}$ being the translation element in spacetime. We then have

$$d\wp(\vee,\vee)d\wp = d\wp(\wedge,\wedge)d\wp, \tag{44}$$

with

$$0 = d\wp(.,.)d\wp = c^2d\tau^2 \pm [c^2dt^2 - \Sigma_i(dx^i)^2], \tag{45}$$

the choice of sign amounting to a choice of signature. This resolves an issue that, as we said, Cartan raised but did not address [30], namely that his theory of moving frames is simply what its name indicates; the motion of points in that theory is represented by the motion of the origin of frames, and not of points viewed as objects unrelated to the frames [14].

To summarize this section up to here, points (a) to (d) have to do directly with limitations of the Kähler calculus. (e) has to do with eliminating a

difficulty introduced in (d) in the process of solving old difficulties. Issues of geometry also are positively impacted by these developments. The ones that follow speak of the potential of the Kähler calculus for the advancement of physics.

(f) Practitioners of Clifford algebras have proposed a variation of them claimed to be specific to quantum field theory, and to which they refer as quantum Clifford algebra, or Clifford algebra of non-symmetric quadratic differential forms [31] (and references therein). In a paper in this journal, it has been shown that such algebras arise naturally in the Kaluza-Klein space to which we have referred above [32].

In another development, Schmeikal has shown how to represent $SU(3)$ in the Clifford algebra of spacetime [33], quarks emerging in families of mutually annulling primitive idempotents. An equivalent family arises in the problem of solving the Kähler equation for systems with a couple of commuting one-parameter symmetries, like in the hydrogen atom. As shown elsewhere [27], simple observation of any of Schmeikal's families and comparison with the one that emerges in solving the hydrogen atom prompts one to think of nucleons as Kähler equation solutions that are not a sum of solutions of Dirac equations. We shall return to this later.

Finally, it is worth mentioning a most intriguing option for unification of gravitation and, at least, electrodynamics. It is made possible by a con-

fluence of the Kähler calculus and geometry. Consider Finsler bundles on the usual pseudo-Riemannian metrics. Regardless of what torsion Ω^{μ} one postulates, only Ω^{0} contributes to the equations of the autoparallels [**?**],[35]. It does so in such way that the pre-Finslerian components of Ω^{0} enter the autoparallels in the same way as the electric and magnetic fields enter the Lorentz force. Those components of Ω^{0} also play, in the linear part of the first Bianchi identity for TP connections, the same role as the electromagnetic form in the homogeneous Maxwell's equations [35]. In our Kaluza-Klein-type space associated with Finslerian TP, that specific part of Ω^{0} is represented by Ω^{4}. The second equation of structure (affine curvature equal to zero) becomes the statement that the metric curvature equals terms that depend on the (con)torsion and derivative thereof. The contribution of Ω^{4} to the Einstein contraction of that equation virtually amounts to the standard electromagnetic energy-momentum tensor [36]. The inhomogeneous Maxwell's equations are still missing in this geometric picture, which unifies the gravitational interaction with whatever interaction(s) is(are) represented by the torsion. Suppose that Maxwell's equations were a degeneration of a geometric Kähler equation resulting from the replacement of its right hand side with a phenomenological alternative [24]. The geometrization of electrodynamics would lie, in the last instance in a totally geometric Kähler equation [37], [35].

5 Killing Symmetries and Constant Differentials: Spinors and Plane Waves.

In this section, we discuss two methods to generate new solutions of the Kähler equation, once a solution has been given. These methods are (a) the action of Killing operators and (b) products by constant differentials. We also make a few considerations meant to replace cumbersome relativistic quantum mechanics theory, like spinors and their transformations.

The term Killing operator is used here to refer to Lie operators that satisfy the Killing relation and leave invariant the input differential form of a given Kähler equation. A most important class of Killing operators are rotation operators. Kähler produced two remarkable results about it. One of them, already mentioned, is that spin is a manifestation of the need to represent the state of an electron by a differential form. A second and related feature is his proposal that the total angular momentum operator appears to assign the electron spin $\hbar$ rather than $(1/2)\hbar$. Since his mathematical argument is incontrovertible and so is the experiment evidences for spin 1/2, a "new picture" for the role of the wave equation emerges, whose potential remains to be examined.

We start with a general remark about rotations in Clifford algebra (In the following, juxtaposition means Clifford product). The action of a rotation on any "Clifford number", whether homogeneous or not, can be given in the

following two ways:

$$u' = e^{\varphi A} \vee u = e^{(\varphi/2)A} \vee u \vee e^{-(\varphi/2)A} \tag{46}$$

where A is a unit bivector in the plane of the rotation. Let us define $\alpha \equiv e^{(\varphi/2)A}$. Under a rotation, a product u ($\equiv xy...z$) of vectors or vector operators becomes u' given by

$$u' = (\alpha x \alpha^{-1})(\alpha y \alpha^{-1}) \vee ... \vee (\alpha z \alpha^{-1}), \tag{47}$$

where we have suppressed several Clifford product symbols to minimize clutter, after adding parentheses for emphasis. The Clifford equation in the rotated frame, $\partial\alpha = a \vee u$, then becomes

$$(\alpha \partial \alpha^{-1})\alpha u = (\alpha a \alpha^{-1}) \vee \alpha u, \tag{48}$$

after multiplication by α on the right. Hence, u maybe viewed as being transformed by the half angle of the rotation, regardless of whether it is spinor or not. What counts is his placement within a differential equation such as the Kähler or Dirac equation.

Next, consider spinors. Corresponding to rotation around an axis, we have the two mutually annulling idempotents

$$\tau^{\pm} = \frac{1}{2}(1 \pm i dx \wedge dy) = \frac{1}{2}(1 \pm i\omega^3) \tag{49}$$

The operators $\tau^{\pm}$ correspond to opposite senses of rotation. They annul each other, and so do the $\epsilon^{\pm}$. The $\tau^{\pm}$ and $\epsilon^{\pm}$ commute with each other. It easily

follows that the complex algebra of scalar-valued differential forms can be viewed as composed of four left ideals:

$$u =^{+} u^{+} \vee \tau^{+} \vee \epsilon^{+} +^{+} u^{-} \vee \tau^{-} \vee \epsilon^{+} +^{-} u^{+} \vee \tau^{+} \vee \epsilon^{-} +^{-} u^{-} \vee \tau^{-} \vee \epsilon^{-} \quad (50)$$

We use the symbol ${}^{\pm}u^{*}$ to refer to both ${}^{\pm}u^{\pm}$ and ${}^{\pm}u^{\mp}$. If u is a solution of the Kähler equation, each of the four ${}^{\pm}u^{*} \vee \tau^{*} \vee \epsilon^{\pm}$ also is a solution. If a system has time-translational and cylindrical symmetry (spherical in particular) each of the solutions corresponds to a solution of the Dirac equation for the same problem. At least such is the case for the hydrogen atom and one may expect similar correspondence in other problems with two "commuting" 1-parameter groups of symmetries. In the Kähler calculus the non-commutativity manifests itself in the impossibility of simultaneously implementing a decomposition into ideals in the sense that we are about to explain.

The decomposition into ideals in (50) is possible because of the commutativity of $\tau^{\pm}$ and $\epsilon^{\pm}$ and that $1 = \epsilon^{+} + \epsilon^{-} = \tau^{+} + \tau^{-}$:

$$u = u \vee (\tau^{+} + \tau^{-}) \vee (\epsilon^{+} + \epsilon^{-}) \quad (51)$$

That defines the ${}^{\pm}u^{*} \vee \tau^{*} \vee \epsilon^{\pm}$, not the ${}^{\pm}u^{*}$ themselves, as is clear for example that

$${}^{+}u^{+} \vee \tau^{+} \vee \epsilon^{+} = ({}^{+}u^{+} + v \vee \tau^{-} \vee \epsilon^{+}) \vee \tau^{+} \vee \epsilon^{+}. \quad (52)$$

Hence, we should more appropriately referred to the ${}^{\pm}u^{*} \vee \tau^{*} \vee \epsilon^{\pm}$ as $u \vee \tau^{*} \vee \epsilon^{\pm}$.

One can define unique coefficients ${}^{\pm}u^*$ as follows. One first proceeds as in obtaining (31). Let a differential form α not depend on dt. We use a basis of polar 1-forms, $(d\rho, d\phi, dz)$ and decompose α as $\alpha' + \alpha'' \vee d\phi$, where neither α' nor α'' contain $d\phi$ as a factor, which defines them uniquely. α can then be expressed as the linear combination of τ^+ and τ^- using that $1 = \tau^+$ and τ^- and $d\phi = i(d\rho/\rho) \vee (\tau^- - \tau^+)$. Hence Eq. (31) can further be expanded to obtain expression (50) with the coefficients ${}^{\pm}u^*$ that this process uniquely defines. They still depend on t and ϕ, dependence which reduces to exponentials on these variables if there is time-translation and cylindrical symmetry.

Consider now how the non-commutativity of rotations around different axes manifests itself in this calculus. For simplicity, let the differential forms be spatial, and the axes of rotation be orthogonal. We substitute $d\phi$ in terms of τ^+ and τ^-. Using $\tan\theta = \rho/z$, we can express $d\theta$ in terms of $d\rho$ and dz. There is the problem, however, that ρ and z do not determine a constant plane, which causes $d\rho$ and dz to not be a constant differential. Hence, one can not deal with two rotations as if one had a time translation and a rotation.

We now turn to the mathematical point made by Kähler with regards to total angular momentum. For this purpose, we first need to consider the spin terms in $X_i u$ (See Eq. (27), namely $(1/2)w_i \vee u - (1/2)u \vee w_i$. Let the

subscript i be 3. We then have

$$ {}^{\pm}u^* \vee \tau^* \vee \epsilon^{\pm} \vee \omega_3 = {}^{\pm} u^* \vee \omega_3 \vee \tau^* \vee \epsilon^{\pm} = -\omega_3^{\pm} \vee^{\pm} u^* \vee \tau^* \vee \epsilon^{\pm} \tag{53} $$

For the first step in (53) we have used that ω_3 commutes with $\tau^{\pm}$ and $\epsilon^{\pm}$. The last step follows from the fact that ω_3 and u^* are respectively proportional to $d\rho \vee d\phi$ and $d\rho \vee dz$ and, therefore,

$$ d\rho \vee dz \vee d\rho \vee d\phi = -d\rho \vee d\phi \vee d\rho \vee dz \tag{54} $$

We thus obtain that

$$ \frac{1}{2}\omega_3 \vee u = -\frac{1}{2} u \vee \omega_3, \tag{55} $$

and that, therefore, the last two terms in (27) are equal.

Consider next the issue of proper values of the spin operator. Since ${}^{\pm}u^* \vee \tau^* \vee \epsilon^{\pm} \vee \omega_3$ equals ${}^{\pm}u^* \vee \tau^* \vee \omega_3 \vee \epsilon^{\pm}$, we have

$$ \tau^{\pm} \vee \omega_3 = \frac{1}{2}(1 \pm i dx \wedge dy) \vee (dx \wedge dy) = \mp i\tau^{\pm} \tag{56} $$

It is then clear that, here as in the standard formalism, we need to multiply by $-i\hbar$ the expression (27) for the operator X_i, so that the proper value of both, orbital angular momentum and spin be real. In particular, the spin term for rotations around the z axes will amount to

$$ \frac{-i\hbar}{2}(\omega_3 \vee u^{\pm} - u^{\pm} \vee \omega_i) = i\hbar u \vee \omega_3 = i\hbar(\mp u^{\pm}) = \pm\hbar u^{\pm} \tag{57} $$

Dirac particles in general, and electrons in particular, would thus appear to acquire spin values of $\pm\hbar$, rather than $\pm\hbar/2$. The way out of this conclusion

could be simply that the Kähler equation does not represent an electron in isolation but rather an electron in an external field, and that the external field constitutes in turn a simplified representation of a second Dirac system (say, the nucleus of the hydrogen atom). In this way, the spin of the two interacting Dirac particles would be represented respectively by each of the two terms in equation (27). This will be clear in later sections from consideration of the standard expanded Hamiltonian for the large components of the wave form. For comparison purposes, let us use the language a little bit freeley, as follows. In the Dirac theory, the large components represent the "electron separated from the positron", the latter being given by the small components. In the Kähler theory, on the other hand, both the large and small components represent the electron, but only the large components obey the Hamiltonian to which we are used. Be as it may, the fine structure of the hydrogen atom obtained with the Kähler equation yields exactly the same result as with the Dirac equation.

Let us proceed with plane waves. We have here four commuting translational symmetries. They allow one to obtain the solutions which are proper functions of energy momentum:

$$u = e^{i(p_l dx^l - Et)} \frac{1}{2^4} (1 \pm dx) \vee (1 \pm dy) \vee (1 \pm dz) \vee (1 \pm idt). \qquad (58)$$

The two signs in $(1 \pm idt)$ correspond to electrons and positrons respectively. Postmultiplying by $(1/2)(1 \pm idx \vee dy)$, one obtains the projection upon proper

states of angular momentum corresponding to opposite senses of rotation.

In the Dirac theory, the spinor entries of plane wave solutions depend on the components of the energy-momentum 4-vector. Here, the dependence on orientation is implicit in the exponential factor at the front. We project upon electron (respectively positron) states by multiplying on the right by ϵ^-(respectively ϵ^+). The information contained in the components of the spinors that reflect the relative orientations of velocity and spin and which is obtained through complicated spinor transformations (even when Clifford algebra formalism is used) will emerge spontaneously through multiplication by $\tau^{\pm}$ on the right. For instance, let z be the spin axis. We project on that axis through multiplication on the right by $(1/2)(1 \pm idx \vee dy)$. This is much simpler than applying rotation operators to the spinors, since these operators are exponentials of terms like $(1/2)(1 \pm idx \vee dy)$. In other words, operating with these constant differentials amounts to working in the Lie algebra of the Lorentz group rather than in the group itself.

This greater simplicity of Kähler's than Dirac's formalism will be encountered time and again as we deal with a variety of issues later in this and, we hope, future papers.

6 Minimal Electromagnetic Coupling: Expansion of the Hamiltonian for Large Components

The Kähler equation with minimal electromagnetic coupling is [3], [4], [5]

$$-i\hbar\partial u = \frac{1}{c}(imc^2 - ec\phi dt + eA_i dx^i) \vee u. \tag{59}$$

Since the mass term dominates at low energies, we write u as

$$u = e^{-imc^2 t/\hbar} R(x,t) \vee \epsilon^-, \tag{60}$$

R depending slowly on time. In the following, we make $\hbar = c = 1$.We substitute (60) in

$$\partial u = dt \vee u_{,t} + dx^i \vee u_{,i}\,, \tag{61}$$

and the resulting expression is replaced in (59). Premultiplying then by $-idt$ and using that $dt \vee dt = -1$, we get

$$\begin{aligned} u_{,t} &= -imu + e^{-imt} R_{,t} \vee \epsilon^- = -dx^i \vee (dt \vee u_{,i}\,) + \\ &\quad + m(dt \vee u) + ieA_i dx^i \vee (dt \vee u) - ie\phi u. \end{aligned} \tag{62}$$

We absorb the factor dt in u and simplify to get

$$R_{,t} = -P \vee \eta R - ie\phi R + im(R - \eta R), \tag{63}$$

where $P \equiv dx^j(-i\partial_j - eA_j)$. Applying η to (63) on the left, and combining the resulting equation with (63) itself, we obtain:

$$\varphi_{,t} = P \vee \chi - ie\phi\varphi, \tag{64}$$

$$\chi_{,t} \;\; = \;\; -P \vee \varphi - ie\phi\chi + 2im\chi, \tag{65}$$

where $\varphi \equiv \frac{1}{2}(R + \eta R)$ and $\chi \equiv \frac{1}{2}(R - \eta R)$. Since m is the dominant energy, Eq. (65) shows that χ is small relative to φ. We define:

$$\chi_1 \equiv -\frac{i}{2m} P \vee \varphi, \qquad\qquad \chi_1' \equiv \chi - \chi_1. \tag{66}$$

Using (65) and (66), we write:

$$\chi_{,t} = -ie\phi\chi + 2im\chi_1', \tag{67}$$

which implies that χ_1' is small relative to χ. We thus have

$$\chi_1' = -\frac{i}{2m}\chi_{,t} + \frac{e\phi}{2m}\chi = \chi_2 + \chi_2', \tag{68}$$

where

$$(\chi_2,\, \chi_2') = -\frac{i}{2m}(\chi_1,\, \chi_1')_{,t} + \frac{e\phi}{2m}(\chi_1,\, \chi_1'), \tag{69}$$

and, therefore

$$\chi = \chi_1 + \chi_2 + \chi_2'. \tag{70}$$

Notice that $\chi_2' \ll \chi_2 \ll \chi_1 \ll \varphi$.

We proceed to write down the equation

$$i\varphi_{,t} = iP \vee Y - ie\phi\varphi, \tag{71}$$

and to develop Eq. (71) with $Y = \chi_1$. Taking into account the first of (66), we further get

$$i\varphi_{,t} = \frac{1}{2m} P \vee (P \vee \varphi) - ie\phi\varphi. \tag{72}$$

Expand $P \vee (P \vee \varphi)$:

$$\begin{aligned} P \vee (P \vee \varphi) &= dx^i \vee dx^j \vee P_i(P_j\varphi) = (dx^i \wedge dx^j) \vee P_i(P_j\varphi) + \\ +\delta^{ij} P_i(P_j\varphi) &= (dx^i \wedge dx^j) \vee [(P_iP_j)\varphi + P_i(P_j^C\varphi)] + P^2\varphi, \end{aligned} \quad (73)$$

where the subscript C denotes that the derivative operators in P_i do not act on P_j, this action already being taken care of by

$$(P_kP_j)\varphi \equiv -i\partial_k(-eA_j) = ieA_{j,k}\,. \quad (74)$$

The antisymmetry of $dx^i \wedge dx^j$ causes the product $(dx^i \wedge dx^j) \vee P_i(P_j^C\varphi)$ to vanish, since $P_i(P_j^C\varphi)$ is symmetric. It also causes the symmetric part of $ieA_{j,k}$ to be ineffectual. Thus, finally,

$$\begin{aligned} P \vee (P \vee \varphi) &= \frac{ie}{2}\sum_{i,j}(dx^i \wedge dx^j) \vee (A_{j,i} - A_{i,j})\varphi + P^2\varphi = \\ &= ieB_k dx^i \vee dx^j \vee \varphi + P^2\varphi. \end{aligned} \quad (75)$$

In the B_k term, one assumes cyclic sum of the three even permutations of $(i = 1,\ j = 2,\ k = 3)$, just once each (thus the disappearance of the factor $1/2$). We thus get, from (72) and (75),

$$i\varphi_{,t} = \frac{1}{2m}P^2\varphi + \frac{ie}{2m}B_k dx^i \vee dx^j \vee \varphi + e\phi\varphi, \quad (76)$$

which is the Pauli equation in terms of differential forms.

The next approximation is now a simple exercise. We readily get:

$$iP \vee \chi_2 = I_1 + I_2 + II, \quad (77)$$

where

$$I_1 \equiv -\frac{i}{4m^2} P \vee (P_{,t} \vee \varphi), \tag{78}$$

$$I_2 \equiv -\frac{i}{4m^2} P \vee (P \vee \varphi_{,t}), \tag{79}$$

and

$$II \equiv \frac{ie}{2m} P \vee (\phi \chi_1) = \frac{ie}{2m} \left(-\frac{i}{2m} \right) P \vee (\phi P \vee \varphi). \tag{80}$$

We rewrite Eq. (78) as

$$I_1 \equiv -\frac{i}{4m^2} (P \vee P_{,t}) \vee \varphi) - \frac{i}{4m^2} P \vee (P_{,t})^C \vee \varphi), \tag{81}$$

where the notation parallels that of Eq. (73). We substitute (64) in (79) with $P \vee \chi$ replaced with $P \vee \chi_1$ in order to stay in the same order as in standard treatments of this problem with the Foldy-Wouthuysen transformation (say as in Bjorken and Drell). Using then (66), we get

$$\begin{aligned} I_2 \quad \equiv \quad & -\frac{i}{4m^2} P \vee \left\{ P \vee \left[-\frac{i}{2m} P \vee (P \vee \varphi) \right] \right\} - \\ & -\frac{i}{4m^2} P \vee \left[P \vee (-ie\phi\varphi) \right]. \end{aligned} \tag{82}$$

The last term in (82) cancels with II. We thus obtain

$$\begin{aligned} I_2 + II \quad \equiv & \\ = \quad & -\frac{1}{8m^3} P \vee \{ P \vee [P \vee (P \vee \phi)] \} - \frac{e}{4m^2} P \vee [(P\phi) \vee \varphi)] = \\ = \quad & \frac{-1}{8m^3} p^4 - \frac{e}{4m^2} \left[P \vee (P\phi) \right] \vee \varphi - \frac{e}{4m^2} P \vee (P\phi)^C \vee \varphi), \end{aligned} \tag{83}$$

where the approximation allowed us to replace the $P^{\vee 4}$ term with $-(8m^3)^{-1}p^4$, and where p^2 is as in non-relativistic quantum mechanics. Using (83) and (81), we finally have

$$iP \vee \chi_2 = \frac{-1}{8m^3}p^4 - \frac{ie}{4m^2}P \vee E^C \vee \varphi - \frac{e}{4m^2}E_{i,j}\,(dx^j \wedge dx^i) \vee \varphi. \qquad (84)$$

The second term on the right hand side of (84) results from combining the $(P_{,t}\,)^C$ term of Eq. (81) with the $(P\phi)^C$ term of (Eq. (83)). Similarly, the third term results from combining the $P_{,t}$ (Eq. (81)) and $P\phi$ terms (Eq. (83)). Once again for the purpose of comparison, we drop higher order terms in further expanding $iP \vee \chi_2$. We thus have

$$-\frac{ie}{4m^2}P \vee E^C \vee \varphi = -\frac{ie}{4m^2}dx^i \wedge dx^j \vee E_j(-i\partial_i)\varphi =$$
$$= -\frac{ie}{4m^2}dx^i \vee dx^j \vee [E_j(-i\partial_i) - E_i(-i\partial_j)]\,\varphi - \frac{ie}{4m^2}E^i p_i. \qquad (85)$$

In more conventional language, these are $\boldsymbol{\sigma} \cdot \mathbf{E} \times \mathbf{p}$ terms. Finally

$$-\frac{e}{4m^2}E_{i,j}\,(dx^j \wedge dx^i) \vee \varphi =$$
$$-\frac{e}{4m^2}E^i{}_{,i} \vee \varphi - \frac{e}{8m^2}(E_{j,i} - E_{i,j}\,)dx^j \wedge dx^i \vee \varphi. \qquad (86)$$

These are the *div* $\mathbf{E}$ and $\boldsymbol{\sigma} \cdot$ *curl* $\mathbf{E}$ terms.

7 Antiparticles, Small Components and "Negative" Energies

In this section, we develop for the large components of positrons the same expanded Hamiltonian equation as for electrons. We then return to the equa-

tions for the electrons of the previous section in order to obtain the Hamiltonian that applies to the small components. We finally show that negative energies constitute a spurious effect arising from the fact that, interpreted from the perspective of the Kähler calculus, the Dirac theory makes the small components of electron wave forms appear as if they belonged to positrons.

In parallel to Eqs. (59)-(61) we now have

$$-i\partial u' = imu' + qAu', \tag{87}$$

$$u' = e^{-imt}S(x,t) \vee \epsilon^+, \tag{88}$$

and

$$\begin{aligned} u'_{,t} &= -imu' + e^{-im't}S_{,t} \vee \epsilon^+ = -dx^i \vee (dt \vee u'_{,i}) + \\ &+m(dt \vee u') + iqA_i dx^i \vee (dt \vee u') - iq\phi u'. \end{aligned} \tag{89}$$

where S depends slowly on t, but not on dt. At this point, the computation changes slightly relative to the previous section in that, whereas we then encountered the product $dt \vee \epsilon^-$ $(= -\epsilon^-)$, we now contend with $dt \vee \epsilon^+$ $(= -\epsilon^+)$. Hence, we have

$$S_{,t} = P \vee \eta S - iq\phi S + im(S + \eta S), \tag{90}$$

$$\eta S_{,t} = -P \vee \eta S - iq\phi \eta S + im(S + \eta S), \tag{91}$$

The even part φ' $(= (S + \eta S)/2)$ is now small. Clearly,

$$\varphi'_{,t} = -P \vee \chi' - iq\phi\varphi' + 2im\varphi', \tag{92}$$

$$\chi'_{,t} \;=\; P \vee \varphi' - iq\phi\chi'. \tag{93}$$

The system (92)-(93) is for (χ', φ') what the system (63)-(64) is for (φ, χ). We have thus shown that the large part, now odd, of the positron has the same Hamiltonian as the large part, then even, of the electron, both in an electromagnetic field.

Let us now deal with the issue of the small components. We return to the equations of the previous section, where we were able to develop an approximating process for obtaining the Hamiltonian for φ thanks to the fact that we could solve (65) approximately for χ, which was then substituted in (64). But, the system of differential equations being what it was, it did not let us proceed similarly to obtain an equation for χ. It is possible, however, to proceed in the following alternative way.

For an electron in a time-homogeneous electromagnetic field, we can write the solution of the Kähler equation, not only as in (60), but also as

$$e^{-imt}R(x,t) = e^{-iEt}\Re(x), \tag{94}$$

where E is the actual energy of the electron in a time-independent electromagnetic field. This readily implies:

$$\left\{ \begin{array}{c} \varphi_{,t} \\ \chi_{,t} \end{array} \right\} = i(m-E) \left\{ \begin{array}{c} \varphi \\ \chi \end{array} \right\}. \tag{95}$$

Substituting the first of (95) in (92) and solving for φ we get

$$\varphi = \mu P \vee \chi \tag{96}$$

where

$$\mu \equiv -\frac{i}{m - E + e\phi} \tag{97}$$

It readily follows that

$$P \vee \varphi = (P\mu) \vee (P \vee \chi) + \mu P \vee (P \vee \chi) \tag{98}$$

and, therefore,

$$i\chi_{,t} = -i\mu P \vee (P \vee \chi) - i(P\mu) \vee (P \vee \chi) + e\phi\chi - 2m\chi, \tag{99}$$

which involves only the small components and is an exact equation for the system of an electron in a time-independent electromagnetic field.

We finally ask whether we can transform the system (64)-(65) into another system where the roles of φ and χ are reversed. In order to achieve this, one has to reverse the signs of the first terms on the right hand sides relative to the left hand sides, which is achieved by replacing t with $t'(=-t)$ and multiplying the equations by -1. One thus obtains

$$\varphi_{,t'} = -P \vee \chi + ie\phi\varphi, \tag{100}$$

$$\chi_{,t'} = P \vee \varphi + ie\phi\chi - 2im\chi \tag{101}$$

In order to restore the sign of the $ie\phi\varphi$ terms, we replace e with e' $(=-e)$. We further replace $-ie'\phi - 2im$ with $-ie'\phi'$. We thus get:

$$\varphi_{,t'} = -P \vee \chi - ie'\phi'\varphi + 2im\varphi, \tag{102}$$

$$\chi_{,t'} = P \vee \varphi - ie'\phi'\chi. \tag{103}$$

These equations are formally identical to Eqs. (64)-(65) with the roles of φ and χ exchanged by time reversal and change conjugation. It is important to remark that the transformation performed does not allow us to follow for this system the process of the previous section. The reason is that $-ie'\phi'$ is of the order of $2im$. The changes made do not affect the fact that χ was small relative to φ in the system (64)-(65); it remains so.

Kähler already showed that electrons and positrons come with the same sign of the energy. We now show why positrons appear with negative energy, a spurious effect that has not impeded the Dirac theory, complemented by hole theory, to yield the right results when doing quantum electrodynamics (QED). The equation $u_{,t} = -imu + e^{-imt}R_{,t} \vee \epsilon^-$ (see (62)) says that rest mass energy is an energy term additional to the terms that appear in $R_{,t}$, which is the reason why it does not appear in the Pauli equation. Let us rewrite the equation (102) as

$$\varphi_{,t'} = -P \vee \chi - i(e'\phi' - 2im)\varphi. \tag{104}$$

If $e'\phi'$ (not small) is taken inadvertently as if it were the real $e'\phi$ (small), we then have the additional contribution $-2m$ to the energy equation in order to handle positrons properly in the Dirac theory and, consequently, in QED based on hole theory.

8 Exact Closed Form of the Minimal-Coupling Electromagnetic Hamiltonian for the Large Components of the Wave Form

Although there is no direct important consequence of obtaining a closed form for the development of the Hamiltonian that we started in Section 6, we proceed to obtain one such expression to illustrate the use of techniques related to the Kähler equation. We have in mind solving other problems with that equation in future papers. We define

$$B \equiv -\frac{i}{2m}\partial_t + \frac{e\phi}{2m} \tag{105}$$

and, for integer $i \geq 2$,

$$\chi_i \equiv B\chi_{i-1} = BB\chi_{i-2} = ... = B^{i-2}\chi_2. \tag{106}$$

For the same range of i's, define further $\chi_i' \equiv \chi_{i-1}' - \chi_i$, and thus

$$\chi_{i-1}' = \chi_i + \chi_i', \tag{107}$$

so that the equations

$$\chi = \chi_1 + \chi_2 + \chi_2' = \sum_{i=1}^{n} \chi_i \ + \chi_n' \tag{108}$$

follow. Define $\chi_0' \equiv \chi$. Thus $\chi_1' \equiv B \vee \chi_0$. We use complete induction to show that $\chi_{i+1}' \equiv B \vee \chi_i'$. Correspondingly, we assume $\chi_i' \equiv B \vee \chi_{i-1}'$. Clearly,

$$\chi_{i+1} + \chi_{i+1}' = \chi_i' = B\chi_{i-1}' = B(\chi_i + \chi_i') = \chi_{i+1} + B\chi_i', \tag{109}$$

and, therefore,

$$\chi'_{i+1} = B\chi'_i \ (= B^{i+1}\chi'_0 = B^{i+1}\chi). \tag{110}$$

Let us develop χ_2 without the approximations used after Eq. (81) in developing $iP \vee \chi_2$. We have:

$$\chi_2 = -\frac{1}{4m^2}(P_{,t} \vee \varphi + P \vee \varphi_{,t}\,) - \frac{ie\phi}{4m^2} P \vee \varphi. \tag{111}$$

The first term on the right of (111), can be further developed as:

$$-\frac{1}{4m^2}\left[-edx^j A_{j,t} \vee \varphi + P \vee (P \vee \chi - ie\phi\varphi)\right]. \tag{112}$$

The ϕ terms in (111) and (112) combine to yield $\frac{1}{4m^2}(-edx^j\phi_{,j}\,)$, so that we finally have:

$$\chi_2 = -\frac{1}{4m^2}\left[e\xi \vee \varphi + P \vee (P \vee \chi)\right], \tag{113}$$

where ξ is $E_j dx^j$. We could now apply B to this relatively compact expression to get χ_3 and so on. But that is not of special interest, once we have shown the way.

We proceed to obtain a closed expression, as per the title of this section. Using the first of (95) in (65) we get

$$\chi = -i\gamma P \vee \varphi, \tag{114}$$

where $\gamma = 1/(E + m - e\phi)$. We substitute back in (65) to obtain

$$-\frac{i}{2m}\chi_{,t} = -iC\gamma P \vee \varphi, \tag{115}$$

where $C \equiv (m-E)/2m$. Through the use of equations (114)-(115), we get, denoting, for brevity $\pi \equiv P \vee \varphi$ and $\lambda \equiv e\phi/2m$,

$$\begin{aligned} iP \vee \chi_1' &= P \vee (C\gamma\pi + \gamma\lambda\pi) = \\ &= C(P\gamma) \vee \pi + C\gamma P \vee \pi + P(\gamma\lambda) \vee (P\varphi) + \gamma\lambda P \vee \pi = \\ &= [(C+\lambda)(P\gamma) + \gamma(P\lambda)] \vee (P\varphi) + (C\lambda)\gamma P \vee \pi, \end{aligned} \tag{116}$$

where. From equation (64) and the second of equations (66), we finally have, restoring the $P \vee \varphi$'s,

$$\begin{aligned} i\varphi_{,t} &= i(\varphi_{,t})_{\text{Pauli}} + iP \vee \chi_1' = \\ &= MP \vee (P \vee \varphi) + N \vee (P \vee \varphi) + e\phi\varphi, \end{aligned} \tag{117}$$

where

$$M \equiv \frac{1}{2m} + (C+\lambda)\gamma, \qquad N \equiv (C+\lambda)P\gamma + \gamma(P\lambda). \tag{118}$$

In order to analyze this result, we consider the term $iP \vee \chi_1'$ that complements the Pauli equation. Equation (68) gives χ_1'. Its first term on the right is $-\frac{i}{2m}\chi_{,t}$. Equation (64) for $\chi_{,t}$ shows th epresence of P in from of φ, which together with the P in front of χ_1' yields two P's in front of φ. This is not the case with terms arising from the second term on (68), $(e\phi/2m)\chi$, which yields only one P acting on φ. Thus, in Eq. (117), contributions to the magnetic moment of the electron come exclusively from the M term. Such

contributions (other than Pauli's) are of too high an order relative to QED predictions. Of course, this is no substitute for QED.

The Lamb shift term is of the type $E^i{}_{,i}$ (see comparison in [8] of the explicit $E^i{}_{,i}$ term obtained in the expansion of the Hamiltonian with the implicit $E^i{}_{,i}$ in the Lamb shift term provided by QED). In order to get the contribution to $E^i{}_{,i}$ from ϕ, one needs two P's acting on it. This is not explicit in Eqs. (117)-(118). But we know that an $E^i{}_{,i}$ term must be hidden on the right hand side of (117), the reason being that we actually obtained one such term in section 6. Equation (94) allowed us to bring together and metamorphose the sum of an infinite number of terms.

The Lamb shift corresponds to the difference in behavior of two specific states at the origin. In the last instance, what matters is the terms that contribute to the specific energy levels. If, for the sake of the sample, we assumed that Eq. (117) happened to contain the anomalous magnetic moment and the Lamb shift effects (it certainly does not), it would take place by virtue of respectively the first and second terms on its right hand side. It would then be advisable to speak of type $P \vee (P \vee \varphi)$ and $P \vee \varphi$ terms, rather than the multiple "effects" represented by Feynman diagrams or combinations thereof.

The preceding considerations would seem to be preposterous given that the anomalous magnetic moment and Lamb shift effects are too large to be contained in Eq. (117). In the next section, we shall argue that one should

not dismiss ab initio the possibility that QED results could be replaced by some other approach with the help of Kähler theory, at least until small components are understood. Our computations show that the Dirac theory does not even reveal the existence of a situation to be explained.

The structural similarity between Eqs. (99) and (117) is to be noticed, abstraction made of the difference in the coefficients. The most important difference is the role of the term $e\phi\varphi$, role played by $e\phi\chi - 2m\chi$. Once again, one may interpret that by saying that, if one insists on incorrectly interpreting χ (i.e. the small components in the case of electrons) as positrons, these appear to be endowed with energy $mc^2 + e\phi - 2mc^2$, i.e. $-mc^2 + e\phi$. The emergence of the negative energy $-mc^2$ is interpreted by Kähler theory as a spurious effect of Dirac's theory.

9 Closing Remarks

The computations of sections 6 to 8 show that Kähler's is an approach to quantum physics far more powerful than Dirac's. Gone is the need for negative energy solutions to represent positrons in pre-QED quantum physics[5]. Gone also is the need for Foldy-Wouthuysen transformations. This calculus thus is advantageous for physicists. It should be clear that doing QED without explicit field quantization can be repeated with this new calculus in approximate parallelism with the approach to quantum electrodynamics in

the first volume of Bjorken and Drell [8]). Work in that direction would be said to constitute "QED based on Kähler". Such course of action, though important, may not even be the most relevant one, as we now explain.

In 1928, simultaneously with the birth of the Dirac's theory and his equation for the electron, Iwanenko and Landau produced an alternative theory based on the use of inhomogeneous antisymmetric tensors in lieu of wave functions. Their work may be viewed as a very crude and ad hoc precursor of the physical part of Kähler's theory [38]. As pointed out by those authors, their theory was more amenable than Dirac's to the treatment of multiparticle systems. That advantage is to be expected for Kähler theory also. In addition, the facts that φ and χ satisfy different differential equations and that the (developed) Hamiltonian that appears to represent the physics is that of φ (not of χ for the electron; not of R; not of u) raise the issues of why this is so, and what does χ represent. Hence the present paper should be viewed as just the beginning of a field of research on Kähler equations, not the last step prior to "QED based on Kähler". The reach of Kähler equations remains to be determined. And there is more.

Work on TP (see [36] and references therein) suggests that the development of Kähler theory may have enormous implications for the philosophy of science, and philosophy itself. For the sake of the argument, let us start with the crude remark that the approach of Heisenberg to quantum mechanics

is operator-based, and that of Schrödinger is based on finding appropriate differential equations (We said crude because he actually found his namesake equation and the one nowadays known as Klein-Gordon's through replacement of physical magnitudes with appropriate operators). In any case, a substantial difference in that general direction has been formalized in the literature: Heisenbergs and Schrödinger pictures.

The first systematic, text-book-level exposition of the principles of quantum mechanics appears to be, happily enough for our purposes, Dirac's classic [23]. His approach with regards to his equation of motion was similar in spirit to Schrödinger's (namely, replace physical magnitudes with corresponding operators). However, his approach is sharply focussed on operators, representations and transformation theory; equations of motion appear almost as an afterthought. It is not a coincidence that QED was born mainly from the efforts of Dirac, on the one hand, and Heisenberg and collaborators on the other. Let us bring now Landau back into our story.

In Landau and Lifchitz's approach to the principles of quantum mechanics [39], there is even less resort to operators, at least implicitly. A professional physicist might easily dismiss these differences as trivial, since all the approaches end up in the same place. The best argument would be the frustration experienced by a student, however bright, asked to learn quantum mechanics by reading [23] and [39] in parallel.

Let us bring now Kähler into the picture and consider the following ordered series: Heisenberg/Dirac, Schrödinger, Landau/Iwanenko/Lifchitz, Kähler. As we move from left to right in this series, the operator flavor decreases and the differential equation flavor increases. In Kähler, the operators emerge from the theory, rather than being used to create it. See our generation of Hamiltonians in sections (6)-(8), as well as the appearance of $dx^j(-i\partial_j - eA_j)$. Contrast that with the form in which corresponding operators were introduced by Dirac [23]. The physical part of Kähler's theory is about a continuous u field whose physical significance is left undefined. Hamiltonians and momenta emerge from the Kähler equation only after one replaces the a in his equation with an input differential form that relates specifically to particles through the mass term.

It first occurred to Einstein to postulate TP for geometrization of the physics. But it was Cartan who made first, correct and easy headway with it. Our work in this regard (see specially [28], [37], [35]) connects Kähler equations whose input differential forms are Clifford valued with the geometry of a Kaluza-Klein type space that we have derived from Finslerian TP. Any additional progress in this direction would make increasingly possible the expansion of the aforementioned series to the right as follows: Heisenberg/Dirac, Schrödinger, Landau/Iwanenko/Lifchitz, Kähler, Cartan/Einstein. The step from Kähler to Cartan/Einstein still remains a huge gap to bridge, which

certainly is not possible if one does not try. Let us provide direction.

The first equation of structure (FES) is the specification of the torsion. In TP, the first Bianchi identity (FBI) states that the exterior covariant derivative of the torsion is zero. Hence it constitutes a *constraint* (specifically a necessary integrability condition), so that a connection with such a torsion may exist. The FBI may also be viewed as a *down payment* on the specification of the torsion, given by the FES, However, such specification can be approached differently. Since the FBI already goes some way towards specifying the torsion, one need only *complement it, i.e. add* to the FBI. That complement is the specification of the interior covariant derivative. Together, they constitute the Kähler derivative, which specifies via integration the differential form of which that derivative is given. Hence the Cartan/Einstein project would meet quantum mechanics if one were to get some kind of Kähler equation canonically determined by the differential invariants of teleparallel structures of some kind. The sought equation should be such that the exterior covariant derivative part of it should become zero when the wave form is meant to represent the torsion in a theory of moving frames (i.e. geometry à la Cartan). This is an approach based on classical techniques, meaning differential equations, which is th eway that Einstein would have wanted it. The promising but still untested possibilities that the Kähler calculus has to offer make unnecessary further advocacy of such a course of action. May this

paper help reduce that gap.

10 Acknowledgements

The author acknowledges interesting conversations with Professors P. Mazur, Z. Oziewicz and A. Staruskewicz on matters related to this paper. Generous funding by PST Associates is deeply appreciated.

References

[1] É. Cartan, Ann. Ec. Normale, **16**, 239 (1899). Reprinted in Œuvres Complètes (CNRS, Paris, 1984). Vol. II, 239-332.

[2] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen (Hamburger Math. Einzelschrifte, 16, 1934).* Reprinted (Chelsea Publishing Company, New York, 1949).

[3] E. Kähler, Abh. Dtsch. Akad. Wiss. Berlin, Kl. für Math. Phys. Tech., No. 4 (1960).

[4] E. Kähler, Abh. Dtsch. Akad. Wiss. Berlin, Kl. für Math. Phys. Tech., No. 1 (1961).

[5] E. Kähler, Rendiconti di Matematica, **21** (1962).

[6] W. Graf, Ann. Inst. Henri Poincaré **19**, 85 (1978).

[7] J. G. Vargas and D. G. Torr, In *Conference Proceedings of Clifford algebras and their Applications in Mathematical Physics, Ixtapa, 1999, Mexico,* edited by R. Ablamowicz and B. Fauser (BirKähleräuser, Boston, 2000), Vol. 1, 135-154.

[8] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw Hill, New York, 1964).

[9] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics (North Holland, Amsterdam, 1982).*

[10] É. Cartan, *Le*çons *sur les invariants intégraux* (Hermann, Paris, 1922).

[11] É. Cartan, Ann. Ec. Norm. **40,** 325 (1923); Reprinted in Œuvres Complètes, Vol. III.

[12] Y. H. Clifton, J. of Math. and Mechanics **16** (1966), 569-576.

[13] R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program (*Springer, New York, 1996).

[14] J. G. Vargas and D. G. Torr, *Balkan Geometry Society Proceedings 13* (Geometry Balkan Press, Bucharest, 2006) pp. 173-182. www.mathem.pub.ro/dept/confer05/M-VAA.PDF.

[15] J. G. Vargas, *Klein Geometries, Spin and Erratum.* Submitted to the Journal of the Balkan Geometry Society www.cartan-einstein-unification.com/pdf/IX.pdf.

[16] H. Flanders, *Differential Forms with Applications to the Physical Sciences* (Dover, New York, 1989).

[17] D. J. Struik, *Lectures on Classical Differential Geometry* (Addison Wesley, Cambridge, MassacHusetts, 1950).

[18] H. Cartan, *Differential Forms*, (Hermann, Paris, 1970) *Reprinted* (Dover, New York, 2006).

[19] A. Lichnerowicz, *Éléments de calcul tensoriel (Jacques Gabay, Paris, 1987).*

[20] W. Rudin, *Principles of Mathematical Analysis* (Mc-Graw Hill, New York, 1976).

[21] W. Slebodzinski, *Formes extérieures et leurs applications, Vol. 2* (Panstwowe Wydawnictwo Naukowe, Warsaw, 1963).

[22] J. G. Vargas and D. G. Torr, J. Math. Phys. **34** (10), 4898 (1993).

[23] P. A. M. Dirac, *The Principles of Quantum Mechanics (Oxford at the Clarendon Press, London, 1966).*

[24] C. A. Mead, *Collective Electrodynamics: Quantum Foundations of Electromagnetism* (The MIT Press, Cambridge, Massachusetts, 2000).

[25] B. Thaller, *The Dirac Equation* (Springer-Verlag, New York, 1992).

[26] J. G. Vargas and D. G. Torr, Found. Phys. **28**, 931 (1998).

[27] J. G. Vargas and D. G. Torr, *New Perspectives on the Kähler Calculus and Wave Functions.* Submitted to the Proceedings of the 7th International Conference on Clifford Algebras. www.cartan-einstein-unification.com/pdf/I.pdf

[28] J. G. Vargas and D. G. Torr, *The Kähler-Dirac Equation with Non-Scalar-Valued Input Differential Form*, Submitted to the Proceedings of the 7th International Conference on Clifford Algebras. www.cartan-einstein-unification.com/pdf/II.pdf

[29] J G. Vargas and D. G. Torr, Found. Phys. **27**, 533 (1997)

[30] É. Cartan, J. Math. pures et appl. **1**, 141 (1922). Reprinted Œuvres Complètes, Vol. III.1 (Éditions du C.N.R.S., Paris, 1983), pp 549-611,

[31] B. Fauser and A. Ablamowicz, in "*Clifford Algebras and their Applications in Mathematical Physics*", edited by B. Fauser and A. Ablamowicz (Birkhäuser, Boston, 2000), Vol. 2, pp.341-365.

[32] J. G. Vargas and D. G. Torr, J. Math. Phys. **43**, 1353 (2002).

[33] B. Schmeikal, in *Conference Proceedings of Clifford Algebras and their Applications in Mathematical Physics*, *Cookeville, 2002, USA*, edited by R. Ablamowicz (Birkhäuser, Boston, 2003), 355-377.

[34] J. G. Vargas and D. G. Torr, Found. Phys. **29**, 1543 (1999).

[35] J. G. Vargas and D. G. Torr, *The Idiosyncrasies of Anticipation in Demiurgic Physical Unification with Teleparallelism.* To be published in the July-August 2006 issue of the International Journal of Computing Anticipatory Systems. www.cartan-einstein-unification.com/pdf/VI.pdf.

[36] J. G. Vargas, *Energy-Momentum Tensors: Perspective from Post-Riemannian Geometry.* An accompanying paper. www.cartan-einstein-unification.com/pdf/VIII.pdf

[37] J. G. Vargas and D. G. Torr, *Balkan Geometry Society Proceedings 13* (Geometry Balkan Press, Bucharest, 2006) pp. 183-191. www.mathem.pub.ro/dept/confer05/M-VAO.PDF

[38] D. Iwanenko and L. D. Landau, Z. Phys. **48**, 340 (1928).

[39] L. Landau and E. Lifchitz, *Mécanique Quantique* (Éditions Mir, Moscow, 1966).